Zeros of Bessel function derivatives

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Abstract

The zeros of Bessel functions and their derivatives play an important role in mathematical physics, and other areas of natural sciences. In this talk our aim is to offer a detailed overview of the results concerning the real zeros of the Bessel functions of the first kind and their derivatives. We prove that for \( \nu > n − 1 \) all zeros of the \( n \)th derivative of Bessel function of the first kind \( J_\nu \) are real. Moreover, we show that the positive zeros of the \( n \)th and \((n + 1)\)th derivative of Bessel function of the first kind \( J_\nu \) are interlacing when \( \nu \geq n \), and \( n \) is a natural number or zero. Our methods include the Weierstrassian representation of the \( n \)th derivative, properties of the Laguerre-Pólya class of entire functions, and the Laguerre inequalities. The main results obtained in this talk generalize and complement some classical results on the zeros of Bessel functions of the first kind. Some conjectures related to Hurwitz theorem on the zeros of Bessel functions are also proposed, which may be of interest for further research.
Definition of Bessel functions of the first and second kind

- By definition, the Bessel functions of the first kind $J_\nu$ is a particular solution of the second order homogeneous Bessel differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.$$
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• The Bessel function of the first kind $J_\nu$ has the power series and Poisson integral representation as follows

$$J_\nu(x) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{1}{2}x\right)^{2n+\nu}}{n! \Gamma(n + \nu + 1)},$$
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$$J_\nu(x) = \frac{2}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(\nu t) dt, \quad \nu > -\frac{1}{2}.$$
The following results are well-known:

- If \( \nu > -1 \), then all zeros of the Bessel function \( J\nu \) are real.

- Theorem (Lommel)

- Theorem (Hurwitz)

- If \( s \) is a nonnegative integer and \( -2s - 2 < \nu < -2s - 1 \), then \( J\nu \) has \( 4s + 2 \) non-real zeros, of which two are purely imaginary.

- If \( s \) is a positive integer and \( -2s - 1 < \nu < -2s \), then \( J\nu \) has \( 4s \) non-real zeros, of which none are purely imaginary.

- It is worth to mention that all zeros of \( J\nu \), except \( x = 0 \) possibly, are simple.
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*The following assertions are true:*

1. If \( \nu > -1 \), then the zeros of \( J_\nu \) are all real.
2. If \( s \) is a nonnegative integer and \( -2s - 2 < \nu < -2s - 1 \), then \( J_\nu \) has \( 4s + 2 \) non-real zeros, of which two are purely imaginary.
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Distribution of real and non-real zeros of Bessel function $J_{\nu}$

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Zeros of Bessel functions via Lommel polynomials

- Hurwitz’s proof

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\[ g_{m,\nu}(x) = \sum_{n=0}^{\infty} C_n^{m-n}(-1)^n \frac{x^n}{\Gamma(\nu+n+1)} \]

\[ \lim_{m \to \infty} g_{m,\nu}(x) = x^{-\nu/2}J_{\nu}(2\sqrt{x}) = \sum_{n \geq 0} (-1)^n \frac{x^n}{n!\Gamma(\nu+n+1)}. \]

3 If \( \nu > -1 \), then \( g_{2m,\nu} \) has \( m \) positive zeros.

4 If \( s \) is a nonnegative integer and \( -2s-2 < \nu < -2s-1 \), then \( g_{2m,\nu} \) has \( m-2s-1 \) positive zeros, 1 negative zero and 2s complex zeros.

5 If \( s \) is a positive integer and \( -2s-1 < \nu < -2s \), then \( g_{2m,\nu} \) has \( m-2s \) positive zeros and 2s complex zeros.

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Zeros of Bessel functions via Lommel polynomials

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Zeros of Bessel functions via Lommel polynomials

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- The modified Lommel polynomial

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g_{2m,\nu}(x) = \sum_{n=0}^{m} C_{m-n}^n \frac{(-1)^n \Gamma(\nu + m - n + 1)x^n}{\Gamma(\nu + n + 1)}
\]

has the following properties:

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\lim_{m \to \infty} \frac{g_{m,\nu}(x)}{\Gamma(\nu + m + 1)} = x^{\frac{\nu}{2}} J_{\nu}(2\sqrt{x}) = \sum_{n \geq 0} \frac{(-1)^n x^n}{n! \Gamma(\nu + n + 1)}. \quad \text{(denoted by } f_{\nu}(x)\text{)}
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2. \[ g_{m+1, \nu}(x) = (\nu + m + 1)g_{m, \nu}(x) - xg_{m-1, \nu}(x). \]

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4. If \( s \) is a nonnegative integer and \( -2s - 2 < \nu < -2s - 1 \), then \( g_{2m,\nu} \) has \( m - 2s - 1 \) positive zeros, 1 negative zero and \( 2s \) complex zeros.

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Zeros of Bessel functions via Lommel polynomials

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The Laguerre polynomial

\[ L^{(n)}_{\alpha}(x) = \sum_{m=0}^{n} C^n_m \frac{(-1)^m x^m}{m!} \]

has the following properties:

1. \( \lim_{n \to \infty} x^{-\nu} L^n_{n} \left( \frac{x}{n} \right) = x^{-\frac{\nu}{2}} J_{\nu}(2\sqrt{x}) = \sum_{n \geq 0} \frac{(-1)^n x^n}{n!(\nu+n+1)}. \) (denoted by \( f_{\nu}(x) \))

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Zeros of Bessel functions via Laguerre polynomials

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2. If \( n > \beta \), \( \beta \) is not an integer, then the polynomial \( L_n^{(-\beta)}(x) \) has \( \lfloor \beta \rfloor \) nonpositive zeros which are all complex, except a single negative one in case \( \lfloor \beta \rfloor \) is odd. See E. Stridsberg\(^5\) and W. Hahn\(^6\).

3. The number of positive zeros of \( L_n^{(\alpha)}(x) \) is \( n \) if \( \alpha > -1 \); it is \( n + \lfloor \alpha \rfloor + 1 \) if \(-n < \alpha < -1\); and it is 0 if \( \alpha < -n \). The number of negative zeros is 0 or 1. See E. Hille and G. Szegő\(^7\).

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\(^5\) E. Stridsberg, Några aritmetiska undersökningar rorande fakulteter och vissa allmännare Koefficientsviter, Not 2, Arkiv Mat., Astronomi och Fysik 13(25) (1918) 1–70.

\(^6\) W. Hahn, Die Nullstellen der Laguerreschen und Hermiteschen Polynôme, Schriften des Mathematischen Seminars und des Instituts für angewandte Mathematik der Universität Berlin 1 (1933) 213–244.

Theorem (Peyerimhoff\textsuperscript{a})


Let $f$ be a real entire function of order $\rho < 1$ with infinitely many zeros $\{a_k\}_{k \in \mathbb{N}}$, and assume that for some nonnegative integer $n$ we have

$a_k \in \{x + iy \mid x \geq 0, \; \text{then } y \neq 0\}$ for $k \in \{1, \ldots, n\}$; and $0 < a_{n+1} < a_{n+2} < \ldots$
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\]
If \( g(x) = \alpha f(x) + xf'(x) \) has exactly one real zero between two zeros \( a_m, a_{m+1} \), \( m \in \{ n+1, n+2, \ldots \} \), then \( g \) has exactly \( n + 1 \) zeros in
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If $g(x) = \alpha f(x) + xf'(x)$ has exactly one real zero between two zeros $a_m, a_{m+1}$, $m \in \{n + 1, n + 2, \ldots\}$, then $g$ has exactly $n + 1$ zeros in $\{x + iy | \text{if } x \geq a_{n+1}, \text{ then } y \neq 0\}$.

The function $f_\nu$. 

Árpád Baricz (Babeș-Bolyai and Óbuda University)
Zeros of Bessel functions via power series

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Zeros of Bessel functions via power series

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Let $f$ be a real entire function of order $\rho < 1$ with infinitely many zeros \$a_k\$ for $k \in \mathbb{N}$, and assume that for some nonnegative integer $n$ we have $a_k \in \{x + iy \mid x \geq 0\}$ for $k \in \{1, \ldots, n\}$; and $0 < a_{n+1} < a_{n+2} < \ldots$.

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Theorem (Peyerimhoff\textsuperscript{a})


If $s \in \mathbb{N}$ and $-s < \nu < -s + 1$, then $f_\nu$ has exactly $s - 1$ zeros that are not positive. Moreover, the function $f_\nu$ has exactly one negative zero if $s = 2k$, no negative zero if $s = 2k - 1$, where $k \in \mathbb{N}$.  

Árpád Baricz  (Babeş-Bolyai and Óbuda University)  
Zeros of Bessel function derivatives  
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To define the notion of the Fourier critical point let $f$ be a real entire function defined in an open interval $(a, b) \subset \mathbb{R}$. Let $l \in \mathbb{N}$ and suppose that $c \in (a, b)$ is a zero of $f^{(l)}(x)$ of multiplicity $m \in \mathbb{N}$, that is, $f^{(l)}(c) = \cdots = f^{(l+m-1)}(c) = 0$ and $f^{(l+m)}(c) \neq 0$. Now, let $k = 0$ if $f^{(l-1)}(c) = 0$, otherwise let

$$k = \begin{cases} 
\frac{1}{2}m, & \text{if } m \text{ is even,} \\
\frac{1}{2}(m + 1), & \text{if } m \text{ is odd and } f^{(l-1)}(c)f^{(l+m)}(c) > 0, \\
\frac{1}{2}(m - 1), & \text{if } m \text{ is odd and } f^{(l-1)}(c)f^{(l+m)}(c) < 0.
\end{cases}$$

We say that $f^{(l)}(x)$ has $k$ critical zeros and $m - k$ noncritical zeros at $x = c$. A point in $(a, b)$ is said to be a Fourier critical point of $f$ if some derivative of $f$ has a critical zero at the point.
Zeros of Bessel functions via Fourier critical points

To define the notion of the Fourier critical point let \( f \) be a real entire function defined in an open interval \((a, b) \subset \mathbb{R}\). Let \( l \in \mathbb{N} \) and suppose that \( c \in (a, b) \) is a zero of \( f^{(l)}(x) \) of multiplicity \( m \in \mathbb{N} \), that is, \( f^{(l)}(c) = \cdots = f^{(l+m-1)}(c) = 0 \) and \( f^{(l+m)}(c) \neq 0 \). Now, let \( k = 0 \) if \( f^{(l-1)}(c) = 0 \), otherwise let

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Definition (Ki, Kim\(^{a}\))


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k = \begin{cases} 
\frac{1}{2} m, & \text{if } m \text{ is even,} \\
\frac{1}{2} (m + 1), & \text{if } m \text{ is odd and } f^{(l-1)}(c)f^{(l+m)}(c) > 0, \\
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Definition (Ki, Kim\(^a\))


We say that \( f^{(l)}(x) \) has \( k \) critical zeros and \( m - k \) noncritical zeros at \( x = c \). A point in \((a, b)\) is said to be a Fourier critical point of \( f \) if some derivative of \( f \) has a critical zero at the point.

• For example, \( \cosh x \) has infinitely many Fourier critical points at the origin with no other Fourier critical points, and the polynomial \( 1 - x^2 + x^8 \) has four Fourier critical points in the whole real axis.
Zeros of Bessel functions via Fourier critical points

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- For example, \( \cosh x \) has infinitely many Fourier critical points at the origin with no other Fourier critical points, and the polynomial \( 1 - x^2 + x^8 \) has four Fourier critical points in the whole real axis. A real analytic function \( f \) has a Fourier critical point if and only if some derivative of \( f \) has more real zeros than guaranteed by Rolle’s theorem.
The zeros $x_1 = -\frac{1}{2} \sqrt[3]{4}$ and $x_2 = -\frac{1}{2} \sqrt[3]{4}$ are critical zeros of $q'(x) = 8x^7 - 2x$; $q''(x) = 56x^6 - 2$ has no critical zeros; and $q'''(x) = 336x^5$ has two critical zeros ($x_3 = x_4 = 0$) and three noncritical zeros; $q^{(l)}$ for $l \in \{4, 5, 6, 7\}$ has no critical zero.
Fourier’s unproved theorem

Let $P$ be a real polynomial of degree $d > 1$. For $n \in \{0, 1, \ldots, d - 1\}$, let $2J_n$ denote the number of nonreal zeros of $P^{(n)}(x)$.
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**Theorem (Fourier unproved theorem)**

Let $f$ be a real entire function and suppose that $f$ can be expressed as the product of a finite or an infinite number of linear factors of the form $(1 - x/\alpha)$, $(1 - x/\beta)$, $(1 - x/\gamma)$, .... Then $f$ has just as many Fourier critical points as couples of non-real zeros.
A real entire function $f$ is of genus $1^*$ if it can be expressed in the form $f(x) = e^{-\alpha x^2}g(x)$, where $\alpha \geq 0$ and $g$ is a real polynomial or a real entire function of genus 0 or 1.
Pólya conjectures on Fourier critical points

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\[ \text{Conjecture (Pólya)} \]

A real integral function of genus 0 has just as many Fourier critical points as couples of imaginary zeros.

Conjecture (Pólya)

If a real integral function of genus 1 has only a finite number of imaginary zeros, it has just as many Fourier critical points as couples of imaginary zeros.

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If a real integral function $f$ of genus 1 has only a finite number of imaginary zeros, its derivatives from a certain one onward, let us say $f^{(n)}$, $f^{(n+1)}$, ... , have only real zeros.
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Solutions of the Fourier-Pólya conjectures

Theorem (Ki, Kim\(^a\))


Let \( f \) be a real entire function that is at most of order 1 and minimum type, let \( b_1, b_2, \ldots \) denote the real zeros of \( f \) that are different from zero, and suppose that
\[
\sum_{|b_j| < r} b_j^{-1} \to \beta \quad \text{as} \quad r \to \infty \quad \text{for some real} \ \beta.
\]
Then for each real constant \( \delta \), the function \( e^{\delta x} f(x) \) has just as many Fourier critical points as couples of non-real zeros.
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for some real $\beta$. Then for each real constant $\delta$, the function $e^{\delta x} f(x)$ has just as many Fourier critical points as couples of non-real zeros.

Theorem (Ki, Kim)

Let $f$ be an even or odd real entire function that is at most of order 2 and minimum type, let $b_1, b_2, \ldots$ denote the real zeros of $f$ that are different from zero, and suppose that

$$\sum_{j>1} b_j^{-2} < \infty.$$  

Then for each nonnegative real constant $\alpha$, the function $e^{-\alpha x^2} f(x)$ has just as many Fourier critical points as couples of non-real zeros.
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Let $f$ be an even or odd real entire function that is at most of order 2 and minimum type, let $b_1, b_2, \ldots$ denote the real zeros of $f$ that are different from zero, and suppose that
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Then for each nonnegative real constant $\alpha$, the function $e^{-\alpha x^2} f(x)$ has just as many Fourier critical points as couples of non-real zeros.

Theorem (Ki, Kim)

Every real entire function of genus 0 has just as many Fourier critical points as couples of non-real zeros.
Consider the auxiliary function $g_{\nu}$, defined by

$$g_{\nu}(x) = \sum_{n \geq 0} \frac{x^n}{\Gamma(n + \nu + 1)n!}.$$ 

We have that

$$J_{\nu}(x) = \left( \frac{x}{2} \right)^\nu g_{\nu} \left( -\frac{x^2}{4} \right).$$
Proof of Hurwitz theorem on the zeros of Bessel functions

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The function $g_{\nu}$ is a real entire function of order $\frac{1}{2}$, so that it is of genus 0.
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Proof of Hurwitz theorem on the zeros of Bessel functions

**Theorem (Ki, Kim*)**


*Let s be a nonnegative integer. Then we have the following:*

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**Árpád Baricz (Babeş-Bolyai and Óbuda University)**

Zeros of Bessel function derivatives

November 29, 2016 14 / 24
Proof of Hurwitz theorem on the zeros of Bessel functions

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Let $s$ be a nonnegative integer. Then we have the following:

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Is it possible to use the above idea of Ki and Kim for other special functions? For example, for Struve functions, Lommel functions, or derivatives of Bessel functions?
Proof of Hurwitz theorem on the zeros of Bessel functions

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In the proofs of the above results two relations were very important:

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g^{(l)}_\nu(x) = g_{\nu+l}(x)
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and

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g^{(l-1)}_\nu(x) - (\nu + l)g^{(l)}_\nu(x) - xg^{(l+1)}_\nu(x) = 0.
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Proof of Hurwitz theorem on the zeros of Bessel functions

Theorem (Ki, Kim\(^a\))


Let \(s\) be a nonnegative integer. Then we have the following:

1. If \(2s − 2 < \nu < −2s − 1\), then \(g_\nu\) has exactly one positive real zero.
2. If \(s\) is a positive integer and \(−2s − 1 < \nu < −2s\), then \(g_\nu(x) \geq 0\) for \(x \geq 0\).

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• Is it possible to use the above idea of Ki and Kim for other special functions? For example, for Struve functions, Lommel functions, or derivatives of Bessel functions?
In view of the results on the zeros of the $n$th derivative of Bessel functions of the first kind, when $n \in \{0, 1, 2, 3\}$, the statements of the following theorem are very natural and somehow expected, they provide the extensions of some classical results on the zeros of Bessel function of the first kind and its derivative of the first order. In the sequel of this talk $n$ and $s$ are from $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$.
Zeros of Bessel function derivatives

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Theorem (Baricz, Kokologiannaki, Pogány$^a$)

$^a$Á. Baricz, C. Kokologiannaki, T.K. Pogány, Zeros of derivatives of Bessel and Struve functions, arXiv.1480790

The following assertions are valid:
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The following assertions are valid:

1. If \( \nu > n - 1 \), then \( x \mapsto J_{\nu}^{(n)}(x) \) has infinitely many zeros, which are all real.
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1. If $\nu > n - 1$, then $x \mapsto J^{(n)}_{\nu}(x)$ has infinitely many zeros, which are all real.
2. If $\nu \geq n$, then the positive zeros of the $n$th and $(n + 1)$th derivative of $J_{\nu}$ are interlacing.
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3. If $\nu > n - 1$, then all zeros of $x \mapsto (n - \nu)J_{\nu}^{(n)}(x) + xJ_{\nu}^{(n+1)}(x)$ are real and interlace with the zeros of $x \mapsto J_{\nu}^{(n)}(x)$. 
Sketch of the proof

1. Weierstrassian canonical representations of the $n$th derivative
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Sketch of the proof

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6. Laguerre theorem on separation of zeros
It is important to mention that the second part of the above theorem in particular reduces to the chains of inequalities

\[ j''', 1 < j''', 2 < j''', 3 < j''', 4 < \ldots, \quad \nu \geq 2 \]

and

\[ j''', 1 < j''', 2 < j''', 3 < j''', 4 < \ldots, \quad \nu \geq 1, \]

where \( j''', n \) and \( j''', n \) denote the \( n \)th positive zero of \( J'' \) and \( J''' \), respectively.
It is important to mention that the second part of the above theorem in particular reduces to the chains of inequalities
\[
\tilde{j}''', 1 < \tilde{j}'', 1 < \tilde{j}''', 2 < \tilde{j}''', 3 < \ldots, \quad \nu \geq 2
\]
and
\[
\tilde{j}''', 1 < \tilde{j}'', 1 < \tilde{j}''', 2 < \tilde{j}''', 3 < \tilde{j}''', 3 < \ldots, \quad \nu \geq 1,
\]
where \(\tilde{j}''', n\) and \(\tilde{j}''', n\) denote the \(n\)th positive zero of \(J''\) and \(J'''\), respectively.

These inequalities complement the well-known ones
\[
\tilde{j}'', 1 < \tilde{j}', 1 < \tilde{j}'', 2 < \tilde{j}'', 2 < \tilde{j}', 3 < \ldots, \quad \nu \geq 0,
\]
where \(\tilde{j}', n\) and \(\tilde{j}'', n\) denote the \(n\)th positive zero of \(J\) and \(J'\), respectively.
It is important to mention that the second part of the above theorem in particular reduces to the chains of inequalities

\[ j'''_{\nu,1} < j''_{\nu,1} < j'''_{\nu,2} < j''_{\nu,2} < j'''_{\nu,3} < j''_{\nu,3} < \ldots, \quad \nu \geq 2 \]

and

\[ j''_{\nu,1} < j'_{\nu,1} < j''_{\nu,2} < j'_{\nu,2} < j''_{\nu,3} < j'_{\nu,3} < \ldots, \quad \nu \geq 1, \]

where \( j''_{\nu,n} \) and \( j'''_{\nu,n} \) denote the \( n \)th positive zero of \( J'_{\nu} \) and \( J'''_{\nu} \), respectively.

These inequalities complement the well-known ones

\[ j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < j_{\nu,3} < \ldots, \quad \nu \geq 0, \]

where \( j_{\nu,n} \) and \( j'_{\nu,n} \) denote the \( n \)th positive zero of \( J_{\nu} \) and \( J'_{\nu} \), respectively.

We also note that the third part of the above theorem is actually a generalization of the well-known result that for \( \nu > -1 \) the zeros of the Bessel functions \( J_{\nu} \) and \( J_{\nu+1} \) are interlacing. Namely, by choosing \( n = 0 \) in the third part we get that the zeros of \( J_{\nu} \) and \( x \mapsto xJ'_{\nu}(x) - \nu J_{\nu}(x) = xJ_{\nu+1}(x) \) are interlacing.
A real entire function $\psi$ belongs to the Laguerre-Pólya class $\mathcal{LP}$ if it can be represented in the form

$$
\psi(x) = cx^m e^{-ax^2 + bx} \prod_{n \geq 1} \left( 1 + \frac{x}{x_n} \right) e^{-\frac{x}{x_n}},
$$

with $c, b, x_n \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}_0$ and $\sum_{n \geq 1} x_n^{-2} < \infty$. We note that the class $\mathcal{LP}$ consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros.
The Laguerre-Pólya class of real entire functions

- A real entire function \( \psi \) belongs to the Laguerre-Pólya class \( \mathcal{LP} \) if it can be represented in the form

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with \( c, b, x_n \in \mathbb{R}, a \geq 0, m \in \mathbb{N}_0 \) and \( \sum_{n \geq 1} x_n^{-2} < \infty \). We note that the class \( \mathcal{LP} \) consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros.

**Theorem (Baricz, Kokologiannaki, Pogány)\(^{\text{a}}\)**

\(^{\text{a}}\)Á. Baricz, C. Kokologiannaki, T.K. Pogány, Zeros of derivatives of Bessel and Struve functions, arXiv.1480790

*If* \( \nu > n - 1 \), *then all the zeros of the Laguerre-type polynomial*

\[
_3F_3\left(-s, \frac{\nu + 1}{2}, \frac{\nu}{2} + 1; \nu + 1, \frac{\nu - n + 1}{2}, \frac{\nu - n}{2} + 1; x\right)
\]

*are real and simple.*
Laguerre-type polynomials

- We note that the denomination Laguerre-type polynomial for the Jensen polynomial appearing in the above theorem is justified by the facts that the case \( n = 0 \) reduces to the well-known generalized Laguerre polynomial \( \genfrac{}{}{0pt}{}{1}{1} \left( -s; \nu + 1; x \right) \) \(^8\), the case \( n = 1 \) one corresponds to the generalized hypergeometric polynomial \( \genfrac{}{}{0pt}{}{2}{2} \left( -s, \frac{\nu}{2} + 1; \nu + 1, \frac{\nu}{2}; x \right) \) which one transforms into the Koornwinder’s generalized Laguerre polynomial \(^9\), while the case \( n = 2 \) is related to the generalized Laguerre polynomial \( \genfrac{}{}{0pt}{}{3}{3} \left( -s, \frac{\nu+1}{2}, \frac{\nu}{2} + 1; \nu + 1, \frac{\nu-1}{2}, \frac{\nu}{2}; x \right) \), considered by Álvarez-Nodarse and Marcellán \(^10\).

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Proof of the reality of zeros of Laguerre-type polynomials

The function

\[ x \mapsto J_{\nu,n}(2\sqrt{x}) = 2^n \Gamma(\nu+1-n) x^{\frac{n-\nu}{2}} J_{\nu}^{(n)}(2\sqrt{x}) = \sum_{m\geq 0} \frac{(-1)^m \Gamma(\nu+2m+1)\Gamma(\nu+1-n)}{m!\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)} x^{m} \]

belongs to the Laguerre-Pólya class \( LP \). Consequently by using the known theorem of Jensen it follows that the Jensen polynomial of \( x \mapsto J_{\nu,n}(2\sqrt{x}) \) has only real zeros.

Now, the Jensen polynomial in the question is

\[ \sum_{m=0}^{s} (-1)^m \binom{s}{m} \frac{\Gamma(\nu+2m+1)\Gamma(\nu+1-n)}{\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)} x^{m}, \]

which after some transformations and in view of the Legendre duplication formula

\[ \Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) \quad (1) \]

can be rewritten as

\[ _{3}F_{3} \left( -s, \frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1, \frac{\nu-n+1}{2}, \frac{\nu-n}{2}+1; x \right). \]

Moreover, according to Csordas and Williamson\(^{11}\) the zeros of the Jensen polynomials are simple, and this completes the proof of the theorem.

Motivated by Hurwitz theorem it is natural to ask about the number of complex zeros of Bessel function derivatives.

\[
\text{Conjecture (Baricz, Kokologiannaki, Pogány)}
\]

Is it true that if \( s \) is a nonnegative integer and \( n \frac{\nu - 2}{2} < \nu < n \frac{\nu - 1}{2} \), then the function \( J_{\nu}(n) \) has \( 4s + 2 \) complex zeros, of which two are purely imaginary?

Is it true that if \( s \) is a positive integer and \( n \frac{\nu - 1}{2} < \nu < n \frac{\nu - 1}{2} \), then the function \( J_{\nu}(n) \) has \( 4s \) complex zeros, of which none are purely imaginary?

As far as we know the methods used in order to prove Hurwitz result on zeros of Bessel functions are not working in the general situation.
• Motivated by Hurwitz theorem it is natural to ask about the number of complex zeros of Bessel function derivatives.

**Conjecture (Baricz, Kokologiannaki, Pogány\(^a\))**

\(^a\)Á. Baricz, C. Kokologiannaki, T.K. Pogány, Zeros of derivatives of Bessel and Struve functions, arXiv.1480790

1. *Is it true that if \(s\) is a nonnegative integer and \(n - 2s - 2 < \nu < n - 2s - 1\), then the function \(J^{(n)}_\nu\) has \(4s + 2\) complex zeros, of which two are purely imaginary?*
Motivated by Hurwitz theorem it is natural to ask about the number of complex zeros of Bessel function derivatives.

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1. Is it true that if $s$ is a nonnegative integer and $n - 2s - 2 < \nu < n - 2s - 1$, then the function $J_{(n)}^\nu$ has $4s + 2$ complex zeros, of which two are purely imaginary?

2. Is it true that if $s$ is a positive integer and $n - 2s - 1 < \nu < n - 2s$, then the function $J_{(n)}^\nu$ has $4s$ complex zeros, of which none are purely imaginary?
Motivated by Hurwitz theorem it is natural to ask about the number of complex zeros of Bessel function derivatives.

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Conjectures on zeros of Bessel function derivatives

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- As far as we know the methods used in order to prove Hurwitz result on zeros of Bessel functions are not working in the general situation.
Conjectures on zeros of Bessel function derivatives

Now, we consider the auxiliary function

\[ f_{\nu, n}(x) = \sum_{m \geq 0} \frac{\Gamma(\nu + 2m + 1)}{\Gamma(\nu + 2m - n + 1)\Gamma(\nu + m + 1)} \frac{x^m}{m!}. \]

This function is a real entire function of growth order \( \frac{1}{2} \) and consequently of genus 0 and has just as many Fourier critical points as couples of nonreal zeros. Now, since \( 2^n x^{n/2} J_{\nu}^{(n)}(2\sqrt{x}) = x^{\nu/2} f_{\nu, n}(-x) \), it follows that when \( \nu > n - 1 \) the function \( f_{\nu, n} \) has no Fourier critical points.
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Is it true that if \( s \) is a nonnegative integer and \( n - 2s - 2 < \nu < n - 2s - 1 \), then the function \( f_{\nu,n} \) has exactly \( s \) Fourier critical points and one positive real zero?
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Conjectures on zeros of Bessel function derivatives

- Now, we consider the auxiliary function

\[ f_{\nu,n}(x) = \sum_{m \geq 0} \frac{\Gamma(\nu + 2m + 1)}{\Gamma(\nu + 2m - n + 1)\Gamma(\nu + m + 1)} \frac{x^m}{m!}. \]

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The monotonicity of the zeros with respect to the order

Recently, Baricz and Szász\textsuperscript{12},

\begin{enumerate}
\item[\textsuperscript{15}] L. Lorch, P. Szegő, Monotonicity of the zeros of the third derivative of Bessel functions, Methods Appl. Anal. 2(1) (1995) 103–111.
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Recently, Baricz and Szász\textsuperscript{12}, and Baricz et al.\textsuperscript{13} found necessary and sufficient conditions on the parameter \( \nu \) such that for \( n \in \{0, 1, 2, 3\} \) the function

\[
z \mapsto 2^\nu \Gamma(\nu - n + 1) z^{\frac{n+2-\nu}{2}} J_\nu^{(n)}(\sqrt{z})
\]

is starlike (maps the open unit disk of the complex plane into a starlike domain) and all of its derivatives are close-to-convex (and hence univalent). In the proofs the key tool was that for fixed \( m \in \mathbb{N} \) and \( n \in \mathbb{N}_0 \) the \( m \)th positive zeros of \( J_\nu^{(n)}(x) \), denoted by \( j_\nu^{(n)}(m) \), are increasing with \( \nu \) on \((n-1, \infty)\), where \( n \in \{0, 1, 2, 3\} \) (see \textsuperscript{14}, \textsuperscript{15}, \textsuperscript{16}, \textsuperscript{17} for more details).

Conjecture

Is it true that \( \nu \mapsto j_\nu^{(n)}(m) \) is increasing on \((n-1, \infty)\) for \( n \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \) fixed?


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**Conjecture**

Is it true that $\nu \mapsto j_{\nu,m}^{(n)}$ is increasing on $(n - 1, \infty)$ for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ fixed?


\textsuperscript{15}L. Lorch, P. Szegő, Monotonicity of the zeros of the third derivative of Bessel functions, Methods Appl. Anal. 2(1) (1995) 103–111.


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Conjecture (Lorch, Muldoon\textsuperscript{a})

\textsuperscript{a}L. Lorch, M. Muldoon, The real zeros of the derivative of cylinder functions of negative order, Methods Appl. Anal. 6(3) (1999) 57–66.

Let $j_{\nu,k}^{(n)}$ be the $k$th positive zero of $J_{\nu}^{(n)}$. 

Another conjecture on zeros of derivatives of Bessel functions

Conjecture (Lorch, Muldoon)


Let $j_{\nu,k}^{(n)}$ be the $k$th positive zero of $J_{\nu}^{(n)}$. Then there exits a unique value $\nu_n$, $n - 3 < \nu_n < n - 2$, such that for $n \in \mathbb{N}$ we have:

1. $j_{\nu,k}^{(n)} < j_{\nu,k+1}^{(n)}$, when $n - 3 < \nu_n < n - 2$,
2. $j_{\nu,1}^{(n)}$ is a double zero of $J_{\nu}^{(n)}$ when $n - 3 < \nu_n < n - 2$,
3. $j_{\nu,1}^{(n)} \rightarrow 0$ and $j_{\nu,2}^{(n)} \rightarrow j_{\nu,n-2}^{(n)}$, as $\nu_n \rightarrow n - 2$.
4. $\nu_n \rightarrow j_{\nu,1}^{(n)}$ is increasing on $(n - 3, \nu_n)$ for $k \in \{2, 3, \ldots\}$,
5. $\nu_n \rightarrow j_{\nu,1}^{(n)}$ is increasing on $(n - 3, \nu_n)$.

Moreover, numerical tests strongly suggest that the sequence $\{\mu_n\}_{n \geq 1}$ defined by

$$
\mu_n = \nu_n - (n - 3),
$$

is decreasing and $\mu_n - \mu_{n+1} \rightarrow 0$ as $n \rightarrow \infty$. 

Árpád Baricz (Babeș-Bolyai and Óbuda University)
Another conjecture on zeros of derivatives of Bessel functions

Conjecture (Lorch, Muldoon)


Let $j^{(n)}_{\nu,k}$ be the $k$th positive zero of $J^{(n)}_{\nu}$. Then there exists a unique value $\nu_n$, $n - 3 < \nu_n < n - 2$, such that for $n \in \mathbb{N}$ we have:

1. $j^{(n)}_{\nu,1} < j^{(n)}_{\nu,2} < j^{(n)}_{\nu,1} < j^{(n)}_{\nu,3}$, $\nu_n < \nu < n - 2$;
Another conjecture on zeros of derivatives of Bessel functions

Conjecture (Lorch, Muldoon\textsuperscript{a})

\textsuperscript{a}L. Lorch, M. Muldoon, The real zeros of the derivative of cylinder functions of negative order, Methods Appl. Anal. 6(3) (1999) 57–66.

Let \( j_{\nu,k}^{(n)} \) be the \( k \)th positive zero of \( J_{\nu}^{(n)} \). Then there exits a unique value \( \nu_n \), \( n - 3 < \nu_n < n - 2 \), such that for \( n \in \mathbb{N} \) we have:

1. \( j_{\nu,1}^{(n)} < j_{\nu,2}^{(n)} < j_{\nu,1}^{(n)} < j_{\nu,3}^{(n)} \), \( \nu_n < \nu < n - 2 \);
2. \( j_{\nu,1}^{(n)} > j_{\nu,1}^{(n)} \), \( n - 3 < \nu < \nu_n \);
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2. $j_{\nu,1}^{(n)} > j_{\nu,1}^{(n)}, n - 3 < \nu < \nu_n$;
3. $j_{\nu,1}^{(n)}$ is a double zero of $J_{\nu}^{(n)}$ when $\nu = \nu_n$;
4. $j_{\nu,1}^{(n)} \rightarrow 0$ and $j_{\nu,2}^{(n)} \rightarrow j_{n-2,1}^{(n)}$ as $\nu \rightarrow n - 2$;
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2. $j_{\nu,1}^{(n)} > j_{\nu,1}$, $n - 3 < \nu < \nu_n$;
3. $j_{\nu,1}^{(n)}$ is a double zero of $J_{\nu}^{(n)}$ when $\nu = \nu_n$;
4. $j_{\nu,1}^{(n)} \to 0$ and $j_{\nu,2}^{(n)} \to j_{n-2,1}^{(n)}$ as $\nu \to n - 2$;
5. $\nu \mapsto j_{\nu,k}^{(n)}$ is increasing on $(n - 3, n - 2)$ for $k \in \{2, 3, \ldots\}$ and $\nu \mapsto j_{\nu,1}^{(n)}$ is increasing on $(n - 3, \nu_n)$.
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Moreover, numerical tests strongly suggest that the sequence $\{\mu_n\}_{n \geq 1}$, defined by $\mu_n = \nu_n - (n - 3)$, is decreasing and $\mu_n - \mu_{n+1} \to 0$ as $n \to \infty$. 