Close-to-convexity of normalized Bessel functions and their derivatives\textsuperscript{1}

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Abstract

In this talk our aim is to present some necessary and sufficient conditions for the starlikeness of Bessel functions of the first kind \( J_\nu \) and their derivatives of the first, second and third order by using a result of Shah and Trimble about transcendental entire functions with univalent derivatives and some Mittag-Leffler expansions for the derivatives of Bessel functions of the first kind, as well as some results on the zeros of these functions. Our aim is also to present necessary and sufficient conditions for the close-to-convexity of some special combinations of Bessel functions of the first kind by using some newly discovered Mittag-Leffler expansions for Bessel functions of the first kind; as well as a necessary and sufficient condition for the close-to-convexity of a cross product of Bessel and modified Bessel functions of the first kind \( J_\nu \) and \( I_\nu \), and their derivatives by using the newly discovered power series and infinite product representation of this cross-product, and a slightly modified version of a result of Lorch on the monotonicity of the zeros of the cross product with respect to the order.
Definition of starlike and convex functions

Let $D(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ be the open disk with center $z_0 \in \mathbb{C}$ and radius $r > 0$ and let us denote the particular disk $D(0, 1)$ by $D$. Moreover, let $S$ be the class of analytic and univalent functions defined in the unit disk $D$, which can be normalized as

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots,$$

that is, $f(0) = f'(0) - 1 = 0$. The class of starlike functions, denoted by $S^*$, is the subclass of $S$ which consists of functions $f$ for which the domain $f(D)$ is starlike with respect to 0. Recall also that a function $f \in S$ belongs to the class $K$ of convex functions if and only if the image domain $f(D)$ is a convex domain in $\mathbb{C}$, that is, the domain $f(D) \subset \mathbb{C}$ contains the entire line segment joining any pair of its points.
The analytic characterizations of $S^*$ and $K$ are the followings

$$S^* = \left\{ f \in S \left| \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \text{ for all } z \in \mathbb{D} \right. \right\},$$

$$K = \left\{ f \in S \left| \operatorname{Re} \left( 1 + zf''(z) f'(z) \right) > 0 \text{ for all } z \in \mathbb{D} \right. \right\}.$$
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Definition of close-to-convex functions

The analytic function \( f : \mathbb{D} \rightarrow \mathbb{C} \) is \textbf{close-to-convex} with respect to the function \( \phi : \mathbb{D} \rightarrow \mathbb{C} \) if \( \Re \left( \frac{f'(z)}{\phi'(z)} \right) > 0 \) for all \( z \in \mathbb{D} \). If there exists a convex function \( \phi : \mathbb{D} \rightarrow \mathbb{C} \) such that \( f \) is close-to-convex with respect to \( \phi \), then we say that \( f \) is close-to-convex.
Definition of close-to-convex functions

The analytic function \( f : \mathbb{D} \rightarrow \mathbb{C} \) is **close-to-convex** with respect to the function \( \phi : \mathbb{D} \rightarrow \mathbb{C} \) if \( \text{Re} \left( \frac{f'(z)}{\phi'(z)} \right) > 0 \) for all \( z \in \mathbb{D} \). If there exists a convex function \( \phi : \mathbb{D} \rightarrow \mathbb{C} \) such that \( f \) is close-to-convex with respect to \( \phi \), then we say that \( f \) is close-to-convex. We note that \( f \) is not required a priori to be univalent, and the associated function \( \phi \) need not be a function belonging to the class \( S \). Moreover, every starlike function is close-to-convex. However, if \( f \) is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function.
Geometric interpretation (Kaplan): Let \( p(z) = \arg f'(z) \) and \( q(z) = \arg \phi'(z) \). Moreover, consider the functions \( P(r, \theta) = p(re^{i\theta}) + \theta \) and \( Q(r, \theta) = q(re^{i\theta}) + \theta \), where \( r < 1 \) and \( \theta \in \mathbb{R} \). A necessary and sufficient condition for \( f \) to be close-to-convex is that \( P(r, \theta_1) - P(r, \theta_2) < \pi \), for \( \theta_1 < \theta_2 \), or equivalently
Geometric interpretation of close-to-convexity

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$$
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) d\theta > -\pi, \quad \text{for} \quad \theta_1 < \theta_2, \quad z = re^{i\theta}.
$$
Geometric interpretation of close-to-convexity

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\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \, d\theta > -\pi, \quad \text{for} \quad \theta_1 < \theta_2, \quad z = re^{i\theta}.
\]

The function \( f \) maps each circle \( z = re^{i\theta} \) onto a simple closed curve whose unit tangent vector \( T = i \exp(iP(r, \theta)) \) either rotates in a counterclockwise direction, as \( \theta \) increases, or else rotates clockwise such that \( \arg T \) never drops to a value \( \pi \) below a previous value, i.e. \( \Delta \arg T \) exceeds \( -\pi \), as \( \theta \) increases.
Hence a subsequence converges uniformly in this domain. By applying the diagonal process in the familiar fashion, we obtain a subsequence of $\phi_n(z)$ which converges uniformly in each circle $|z| < \rho < 1$ and hence has as limit a unique function $\phi(z)$, analytic for $|z| < 1$. Since the $\phi(z)$ are schlicht and convex, $\phi(z)$ must also be so. Since (27) holds for $\rho = \rho_n$, we conclude that

$$\text{Re} \left[ \frac{f'(z)}{\phi'(z)} \right] > 0 \text{ for } |z| < 1;$$

i.e., $f(z)$ is close-to-convex.

4. Geometric interpretation. The condition (16) or its equivalent, condition (16), has the following geometric meaning: $w = f(z)$ maps each circle $z = r e^{i\theta}$ (fixed and $r < 1$) onto a simple closed curve whose unit tangent vector $T = i \exp [i\Phi(r, \theta)]$ either rotates in a counterclockwise direction, as $\theta$ increases, or else rotates clockwise in such a manner that $\arctan \frac{\Phi}{\pi}$ never drops to a value $\frac{\pi}{2}$ radians below a previous value; i.e., $\Delta \arg T$ exceeds $-\pi$, as $\theta$ increases. This is illustrated in Fig. 1. Here $\arg T_2 - \arg T_1$ is only slightly greater than $-\pi$. Thus such a "hairpin turn" is permitted, provided one does not make a complete reversal of direction.

5. Extremal aspects. For each function $f(z)$, analytic and with non-vanishing derivative for $|z| < 1$, we make the definition:

$$c[f] = \text{g.l.b. \;} [\text{ l.u.b. } |\arctan f'(z) - \arctan \phi'(z)|],$$

where $\phi$ ranges over the class of all convex schlicht functions for $|z| < 1$; for each $z$, the arguments of $f'$ and $\phi'$ are to be chosen to give the absolute value of the difference its smallest value. In general, $c[f] \leq \pi$. If $c[f] < \pi$, then as in the preceding section one can compute $c[f]$ by restricting $\phi$ by the conditions (25); the restricted family is normal and accordingly there exists a convex $\phi$ such that

$$|\arctan f'(z) - \arctan \phi'(z)| \leq c[f], \quad |z| < 1;$$

c[f] is the smallest constant for which such a $\phi$ can be found. The function $\phi$ in (30) can be termed a "best convex approximation to $f(z)$".

If $c[f] = 0$, then $f$ must itself be convex; if $c[f] \leq \frac{1}{2} \pi$, then $f$ is close-to-convex.

The constant $c[f]$ and a corresponding extremal $\phi$ satisfying (30) can be found directly by the procedure of Section 3. We introduce the function $P(r, \theta) = \arctan f'(z) + \theta$. Then

$$c[f] = \min \left( \frac{1}{2} \text{ l.u.b. } [P(r, \theta_1) - P(r, \theta_2)], \pi \right),$$
(Alexander) The analytic function $f : \mathbb{D} \to \mathbb{C}$ is convex if and only if $zf'$ is starlike.
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(Ozaki) If the function $z \mapsto f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ is analytic in $\mathbb{D}$ and if $1 \geq 2a_2 \geq \ldots \geq na_n \geq \ldots \geq 0$ or $1 \leq 2a_2 \leq \ldots \leq na_n \leq \ldots \leq 2$, then $f$ is close-to-convex with respect to $z \mapsto -\log(1-z)$. 
(Alexander) The analytic function $f : \mathbb{D} \to \mathbb{C}$ is convex if and only if $zf'$ is starlike.

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(Ozaki) If the function $z \mapsto f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ is analytic in $\mathbb{D}$ and if $1 \geq 2a_2 \geq \ldots \geq na_n \geq \ldots \geq 0$ or $1 \leq 2a_2 \leq \ldots \leq na_n \leq \ldots \leq 2$, then $f$ is close-to-convex with respect to $z \mapsto -\log(1 - z)$. 
Lemma (Shah, Trimble)

Let $f : \mathbb{D} \to \mathbb{C}$ be a transcendental entire function of the form

$$f(z) = z \prod_{n \geq 1} \left(1 - \frac{z}{z_n}\right),$$

where all zeros $z_n$ of $f$ have the same argument and satisfy $|z_n| > 1$. If $f$ is univalent in $\mathbb{D}$, then

$$\sum_{n \geq 1} \frac{1}{|z_n| - 1} \leq 1. \quad (1)$$

In fact (1) holds if and only if $f$ is starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex there.
Theorem (Baricz, Çağlar, Deniz)

The function

\[ z \mapsto 2^{\nu} \Gamma(\nu) z^{\frac{3}{2}} - \nu J_\nu'(\sqrt{z}) \]

is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^* \), where \( \nu^* \simeq 0.7022 \ldots \) is the unique root on \((0, \infty)\) of the equation

\[
(2\nu - 1)J_\nu(1) + (\nu - 2)J_{\nu+1}(1) = 0.
\]
Theorem (Baricz, Çağlar, Deniz)

The function

\[ z \mapsto 2^\nu \Gamma(\nu) z^{\frac{3}{2} - \frac{\nu}{2}} J'_\nu(\sqrt{z}) \]

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is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx 1.9052 \ldots \) is the unique root on \((1, \infty)\) of the equation

\[ (2\nu^2 - 2\nu - 3)J_{\nu}(1) = (\nu^2 + \nu - 3)J_{\nu+1}(1). \]
Theorem (Baricz, Çağlar, Deniz)

The function

\[ z \mapsto 2^\nu \Gamma(\nu - 2)z^{\frac{5}{2} - \frac{\nu}{2}} J'''(\sqrt{z}) \]

is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx 3.077 \ldots \) is the unique root on \( (2, \infty) \) of the equation

\[ (2\nu^3 - 7\nu^2 + 3)J_\nu(1) + (\nu^3 + \nu^2 + \nu - 1)J_{\nu+1}(1) = 0. \]
Theorem (Baricz, Çağlar, Deniz)

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\[ (2\nu^3 - 7\nu^2 + 3)J_{\nu}(1) + (\nu^3 + \nu^2 + \nu - 1)J_{\nu+1}(1) = 0. \]

The next result is a common generalization of the above first two Theorems.
Theorem (Baricz, Çağlar, Deniz)

The function
\[ z \mapsto 2^\nu \Gamma(\nu - 2) z^{\frac{5}{2} - \frac{\nu}{2}} J''''(\sqrt{z}) \]
is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx 3.077 \ldots \) is the unique root on \((2, \infty)\) of the equation
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(2\nu^3 - 7\nu^2 + 3)J_\nu(1) + (\nu^3 + \nu^2 + \nu - 1)J_{\nu+1}(1) = 0.
\]

The next result is a common generalization of the above first two Theorems.

Theorem (Baricz, Çağlar, Deniz)

Let \( a, b, c \in \mathbb{R} \) such that \( c = 0 \) and \( b \neq a \) or \( c > 0 \) and \( b > a \). Moreover, suppose that \( \nu \geq \overline{\nu} \), where \( \overline{\nu} = \max\{0, \nu_0\} \) and \( \nu_0 \) is the largest root of the quadratic \( Q(\nu) = a\nu(\nu - 1) + b\nu + c \). Assume also that the following inequalities are valid
\[
Q(\nu) + 4a\nu + 2a + 2b > 0, \quad (4\nu + 3)Q(\nu) > 4a\nu + 2a + 2b. \tag{2}
\]

Then \( z \mapsto 2^\nu Q^{-1}(\nu) \Gamma(\nu + 1) z^{1 - \frac{\nu}{2}} (azJ''(\sqrt{z}) + b\sqrt{z}J'_{\nu}(\sqrt{z}) + cJ_{\nu}(\sqrt{z})) \) is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^\circ \), where \( \nu^\circ \) is the unique root on \((\overline{\nu}, \infty)\) of the transcendent equation
\[
(2a\nu^2 - 2a\nu + 2b\nu - 3a - b + 2c)J_{\nu}(1) = (a\nu^2 + a\nu - b\nu - 3a + 2b + c)J_{\nu+1}(1).
\]
It is worth to mention that when \( b = c = 0 \) and \( a = 1 \), then the fourth Theorem reduces to the second Theorem. In this case \( \nu = 1 \), \( \nu^\circ \) becomes \( \nu^* \) and the inequalities (2) become \( \nu^2 + 3\nu + 2 > 0 \), and \( 4\nu^3 - \nu^2 - 7\nu - 2 > 0 \). These inequalities give \( \nu > -1 \) and \( \nu > 1.5687\ldots \), which are certainly satisfied for \( \nu > \nu^* \).
► It is worth to mention that when $b = c = 0$ and $a = 1$, then the fourth Theorem reduces to the second Theorem. In this case $\bar{\nu} = 1$, $\nu^\circ$ becomes $\nu^*$ and the inequalities (2) become $\nu^2 + 3\nu + 2 > 0$, and $4\nu^3 - \nu^2 - 7\nu - 2 > 0$. These inequalities give $\nu > -1$ and $\nu > 1.5687\ldots$, which are certainly satisfied for $\nu > \nu^*$.

► Similarly, we note that when $a = c = 0$ and $b = 1$, then the fourth Theorem reduces to the first Theorem. In this case $\bar{\nu} = 0$, $\nu^\circ$ becomes $\dot{\nu}$ and the inequalities (2) become $\nu + 2 > 0$, and $4\nu^2 + 3\nu - 2 > 0$. These inequalities give $\nu > -2$ and $\nu > 0.4253\ldots$, which are certainly satisfied for $\nu > \dot{\nu}$. 
Proof of the first Theorem

Let us denote by $j'_{\nu,n}$ the $n$th positive zero of the function $J'_\nu$. 

By using the infinite product representation 

$J'_\nu(z) = \left(\frac{z}{2}\right)^{-\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) \prod_{n \geq 1} \left(1 - \left(\frac{z}{2}\right) j'_{\nu,n}^2\right)$

it follows that 

$2\nu \Gamma\left(\frac{\nu}{2}\right) z^2 - \nu^2 J''_\nu\left(\sqrt{z}\right) = \prod_{n \geq 1} \left(1 - \left(\frac{z}{2}\right) j'_{\nu,n}^2\right)$

and 

$-\frac{1}{2} \left(1 - \nu + J''_\nu(z) J'_\nu(z)\right) = \sum_{n \geq 1} z^2 j'_{\nu,n}^2 - z^2$.

On the other hand, we know that $\nu \mapsto j'_{\nu,n}$ is increasing on $(0, \infty)$ for each $n \in \mathbb{N}$ fixed, and thus the function $\nu \mapsto \sum_{n \geq 1} z^2 j'_{\nu,n}^2 - \frac{1}{2} \left(1 - \nu + J''_\nu(1) J'_\nu(1)\right)$ is decreasing on $(0, \infty)$. 

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Proof of the first Theorem

Let us denote by $j'_{\nu,n}$ the $n$th positive zero of the function $J'_{\nu}$. By using the infinite product representation

$$J'_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu-1}}{2\Gamma(\nu)} \prod_{n \geq 1} \left(1 - \frac{z^2}{j'_{\nu,n}^2}\right)$$

and

$$\frac{-1}{2} \left(1 + j'_{\nu,n} \frac{J''_{\nu}(1)}{J'_{\nu}(1)}\right) = \sum_{n \geq 1} z^2.$$
Proof of the first Theorem

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$$J'_\nu(z) = \left(\frac{\frac{z}{2}}{2\Gamma(\nu)}\right)^{\nu-1} \prod_{n \geq 1} \left(1 - \frac{z^2}{j'^2_{\nu,n}}\right)$$

it follows that

$$2^{\nu} \Gamma(\nu) z^{\frac{3}{2} - \frac{\nu}{2}} J''_{\nu}(\sqrt{z}) = z \prod_{n \geq 1} \left(1 - \frac{z}{j''_{\nu,n}}\right)$$
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and

$$-\frac{1}{2} \left(1 - \nu + \frac{z J'''_{\nu}(z)}{J'_\nu(z)}\right) = \sum_{n\geq 1} \frac{z^2}{j''_{\nu,n} - z^2}.$$
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On the other hand, we know that $\nu \mapsto j'_{\nu,n}$ is increasing on $(0, \infty)$ for each $n \in \mathbb{N}$ fixed,
Proof of the first Theorem

Let us denote by \( j'_{\nu,n} \) the \( n \)th positive zero of the function \( J'_{\nu} \). By using the infinite product representation

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J'_{\nu}(z) = \frac{(\frac{z}{2})^{\nu-1}}{2\Gamma(\nu)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{j'^2_{\nu,n}} \right)
\]

it follows that

\[
2^\nu \Gamma(\nu) z^{\frac{3}{2} - \frac{\nu}{2}} J''_{\nu}(\sqrt{z}) = z \prod_{n \geq 1} \left( 1 - \frac{z}{j''_{\nu,n}} \right)
\]

and

\[
-\frac{1}{2} \left( 1 - \nu + \frac{zJ''_{\nu}(z)}{J'_{\nu}(z)} \right) = \sum_{n \geq 1} \frac{z^2}{j''_{\nu,n} - z^2}.
\]

On the other hand, we know that \( \nu \mapsto j'_{\nu,n} \) is increasing on \( (0, \infty) \) for each \( n \in \mathbb{N} \) fixed, and thus the function

\[
\nu \mapsto \sum_{n \geq 1} \frac{1}{j''_{\nu,n} - 1} = -\frac{1}{2} \left( 1 - \nu + \frac{J''_{\nu}(1)}{J'_{\nu}(1)} \right)
\]

is decreasing on \( (0, \infty) \).
Proof of the first Theorem

Consequently, we have that the inequality

\[ \sum_{n \geq 1} \frac{1}{j^{2,\nu,n} - 1} \leq 1 \]

is valid if and only if \( \nu \geq \dot{\nu} \),
Proof of the first Theorem

Consequently, we have that the inequality

\[ \sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} \leq 1 \]

is valid if and only if \( \nu \geq \nu^* \), where \( \nu^* \) is the unique root on \((0, \infty)\) of the equation

\[ \sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} = 1 \iff (3 - \nu)J_\nu'(1) + J_\nu''(1) = 0. \quad (3) \]
Proof of the first Theorem

Consequently, we have that the inequality

$$\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} \leq 1$$

is valid if and only if $\nu \geq \hat{\nu},$ where $\hat{\nu}$ is the unique root on $(0, \infty)$ of the equation

$$\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} = 1 \iff (3 - \nu)J_\nu'(1) + J''_\nu(1) = 0.$$  \hfill (3)

Since $J_\nu$ satisfies the Bessel differential equation, it follows that

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0,$$

and then

$$J''_\nu(1) = (\nu^2 - 1)J_\nu(1) - J'_\nu(1) = (\nu^2 - \nu - 1)J_\nu(1) + J_{\nu+1}(1),$$
Proof of the first Theorem

Consequently, we have that the inequality

$$\sum_{n \geq 1} \frac{1}{j''_{\nu,n} - 1} \leq 1$$

is valid if and only if $\nu \geq \dot{\nu}$, where $\dot{\nu}$ is the unique root on $(0, \infty)$ of the equation

$$\sum_{n \geq 1} \frac{1}{j''_{\nu,n} - 1} = 1 \iff (3 - \nu)J'_\nu(1) + J''_\nu(1) = 0. \tag{3}$$

Since $J_\nu$ satisfies the Bessel differential equation, it follows that

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0,$$

and then

$$J''_\nu(1) = (\nu^2 - 1)J_\nu(1) - J'_\nu(1) = (\nu^2 - \nu - 1)J_\nu(1) + J_{\nu+1}(1),$$

where we used the recurrence relation $zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$.
Proof of the first Theorem

Consequently, we have that the inequality
\[
\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} \leq 1
\]
is valid if and only if \( \nu \geq \tilde{\nu} \), where \( \tilde{\nu} \) is the unique root on \((0, \infty)\) of the equation
\[
\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 - 1} = 1 \iff (3 - \nu)J'_{\nu}(1) + J''_{\nu}(1) = 0. \tag{3}
\]

Since \( J_\nu \) satisfies the Bessel differential equation, it follows that
\[
z^2 J''_{\nu}(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0,
\]
and then
\[
J''_{\nu}(1) = (\nu^2 - 1)J_\nu(1) - J'_\nu(1) = (\nu^2 - \nu - 1)J_\nu(1) + J_{\nu+1}(1),
\]
where we used the recurrence relation \( zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z) \). Consequently, the equation (3) is equivalent to
\[
(2\nu - 1)J_\nu(1) + (\nu - 2)J_{\nu+1}(1) = 0.
\]
Proof of the first Theorem

Consequently, we have that the inequality

\[ \sum_{n \geq 1} \frac{1}{j'_{\nu,n}^2 - 1} \leq 1 \]

is valid if and only if \( \nu \geq \hat{\nu} \), where \( \hat{\nu} \) is the unique root on \((0, \infty)\) of the equation

\[ \sum_{n \geq 1} \frac{1}{j'_{\nu,n}^2 - 1} = 1 \iff (3 - \nu)J'_\nu(1) + J''_\nu(1) = 0. \tag{3} \]

Since \( J_\nu \) satisfies the Bessel differential equation, it follows that

\[ z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0, \]

and then

\[ J''_\nu(1) = (\nu^2 - 1)J_\nu(1) - J'_\nu(1) = (\nu^2 - \nu - 1)J_\nu(1) + J_{\nu+1}(1), \]

where we used the recurrence relation \( zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z) \). Consequently, the equation (3) is equivalent to

\[ (2\nu - 1)J_\nu(1) + (\nu - 2)J_{\nu+1}(1) = 0. \]

Now, applying the inequality \( j'_{\nu,1}^2 > 4\nu(\nu + 1)/(\nu + 2) \), where \( \nu > 0 \), it follows that for \( n \in \{2, 3, \ldots\} \) we have \( j'_{\nu,n} > \ldots > j'_{\nu,1} > 1 \) if \( \nu > (-3 + \sqrt{41})/8 \approx 0.4253 \ldots \). Thus, applying the Lemma of Shah and Trimble the assertion of the theorem follows.
Now, consider the function $f_{\nu} : \mathbb{D} \rightarrow \mathbb{C}$, defined by

$$f_{\nu}(z) = 2^{\nu} \Gamma(\nu + 1) z^{1 - \frac{\nu}{2}} J_{\nu}(\sqrt{z}) = \sum_{n \geq 0} \frac{(-1)^n \Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}.$$
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Very recently by using a result of Shah and Trimble it was proved that the function $f_{\nu}$ and all of its derivatives are convex in $\mathbb{D}$ if and only if $\nu \geq \nu_*$, where $\nu_* \simeq -0.1438 \ldots$ is the unique root of the equation

$$3J_\nu(1) + 2(\nu - 2)J_{\nu+1}(1) = 0$$

on $(-1, \infty)$. 
Now, consider the function \( f_{\nu} : \mathbb{D} \rightarrow \mathbb{C} \), defined by

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Very recently by using a result of Shah and Trimble it was proved that the function \( f_{\nu} \) and all of its derivatives are convex in \( \mathbb{D} \) if and only if \( \nu \geq \nu_{*} \), where \( \nu_{*} \approx -0.1438 \ldots \) is the unique root of the equation

\[
3J_{\nu}(1) + 2(\nu - 2) J_{\nu+1}(1) = 0
\]
on \((-1, \infty)\). We note that in view of the Alexander’s duality theorem the first part of the above result is equivalent to the fact that the function

\[
z \mapsto q_{\nu}(z) = zf'_{\nu}(z) = 2^{\nu - 1} \Gamma(\nu + 1) z^{1 - \frac{\nu}{2}} ((2 - \nu)J_{\nu}(\sqrt{z}) + \sqrt{z}J'_{\nu}(\sqrt{z}))
\]

is starlike in \( \mathbb{D} \) if and only if \( \nu \geq \nu_{*} \).
Now, consider the function $f_\nu : \mathbb{D} \rightarrow \mathbb{C}$, defined by

$$f_\nu(z) = 2^\nu \Gamma(\nu + 1)z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}) = \sum_{n \geq 0} \frac{(-1)^n \Gamma(\nu + 1)z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}.$$ 

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$$= \sum_{n \geq 0} \frac{(-1)^n (n+1) \Gamma(\nu + 1)z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}$$

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Now, consider the function $f_{\nu} : \mathbb{D} \to \mathbb{C}$, defined by

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is starlike in $\mathbb{D}$ if and only if $\nu \geq \nu_*$. We would like to point out that actually we have the following stronger result.

**Theorem (Baricz, Deniz, Yağmur)**

The function $q_{\nu}$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu_*$, where $\nu_* \simeq -0.1438 \ldots$ is the unique root of the transcendent equation $3J_{\nu}(1) + 2(\nu - 2)J_{\nu+1}(1) = 0$ on $(-1, \infty)$. 

Árpád Baricz (Babeş-Bolyai University)
Consider the normalized Dini function $r_\nu : \mathbb{D} \to \mathbb{C}$, defined by

$$r_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\frac{\nu}{2}} ((1 - \nu) J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z})) = \sum_{n \geq 0} \frac{(-1)^n (2n + 1) \Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}.$$
Consider the normalized Dini function \( r_\nu : \mathbb{D} \to \mathbb{C} \), defined by

\[
r_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1 - \frac{\nu}{2}} \left( (1 - \nu)J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z}) \right) = \sum_{n \geq 0} \frac{(-1)^n(2n + 1)\Gamma(\nu + 1) z^{n + 1}}{4^n n! \Gamma(\nu + n + 1)}
\]

Theorem (Baricz, Deniz, Yağmur)

The function \( r_\nu \) is starlike and all of its derivatives are close-to-convex (and hence univalent) in \( \mathbb{D} \) if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx 0.3062 \ldots \) is the unique root of the transcendent equation \( J_\nu(1) - (3 - 2\nu)J_{\nu+1}(1) = 0 \) on \( (0, \infty) \).
Consider the normalized Dini function $r_\nu : D \to \mathbb{C}$, defined by

$$r_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\frac{\nu}{2}} ((1 - \nu) J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z})) = \sum_{n \geq 0} \frac{(-1)^n (2n + 1) \Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}.$$

**Theorem (Baricz, Deniz, Yağmur)**

The function $r_\nu$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $D$ if and only if $\nu \geq \nu^*$, where $\nu^* \simeq 0.3062 \ldots$ is the unique root of the transcendent equation $J_\nu(1) - (3 - 2\nu) J_{\nu+1}(1) = 0$ on $(0, \infty)$.

Now, let us consider the function $w_{\alpha,\nu} : D \to \mathbb{C}$, defined by

$$w_{\alpha,\nu}(z) = \frac{2^\nu z^\alpha}{\alpha z^\frac{\nu}{2}} \Gamma(\nu+1) \left((\alpha - \nu) J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z})\right) = \sum_{n \geq 0} \frac{(-1)^n (2n + \alpha) \Gamma(\nu + 1) z^{n+1}}{\alpha \cdot 4^n n! \Gamma(n + \nu + 1)}.$$

**Theorem (Baricz, Deniz, Yağmur)**

Let $\nu > \frac{-3}{4}$ and $\alpha \geq \frac{2}{4} \nu + \frac{3}{2}$. The function $w_{\alpha,\nu}$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $D$ if and only if $\nu \geq \nu_{\alpha}$, where $\nu_{\alpha}$ is the unique root of the equation $(2\alpha - 1) J_\nu(1) - (\alpha - 2\nu) J_{\nu+1}(1) = 0$ on $(-\frac{3}{4}, \infty)$. 
Consider the normalized Dini function $r_\nu : \mathbb{D} \rightarrow \mathbb{C}$, defined by

$$r_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\frac{\nu}{2}} ((1 - \nu) J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z})) = \sum_{n \geq 0} (-1)^n (2n + 1) \frac{n!}{\Gamma(\nu + n + 1)} \frac{\Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(\nu + n + 1)}.$$

Theorem (Baricz, Deniz, Yağmur)

The function $r_\nu$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu^*$, where $\nu^* \simeq 0.3062 \ldots$ is the unique root of the transcendent equation $J_\nu(1) - (3 - 2\nu) J_{\nu+1}(1) = 0$ on $(0, \infty)$.

Now, let us consider the function $w_{\alpha, \nu} : \mathbb{D} \rightarrow \mathbb{C}$, defined by

$$w_{\alpha, \nu}(z) = \frac{2^\nu}{\alpha z^{\frac{\nu}{2}}} \Gamma(\nu + 1) ((\alpha - \nu) J_\nu(\sqrt{z}) + \sqrt{z} J'_\nu(\sqrt{z})) = \sum_{n \geq 0} (-1)^n \frac{2n + \alpha}{\alpha} \frac{n!}{\Gamma(n + \nu + 1)} \frac{\Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(n + \nu + 1)}.$$

The following sharp result is a common generalization of the above last two Theorems.
Close-to-convexity of the normalized Dini functions

Consider the normalized Dini function $r_\nu : \mathbb{D} \to \mathbb{C}$, defined by
\[
r_\nu(z) = 2\nu \Gamma(\nu + 1)z^{1-\nu/2} \left((1-\nu)J_\nu(\sqrt{z}) + \sqrt{z}J'_\nu(\sqrt{z})\right) = \sum_{n \geq 0} \frac{(-1)^n(2n + 1)\Gamma(\nu + 1)z^{n+1}}{4^n n!\Gamma(\nu + n + 1)}.
\]

Theorem (Baricz, Deniz, Yağmur)

The function $r_\nu$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu^*$, where $\nu^* \simeq 0.3062 \ldots$ is the unique root of the transcendent equation $J_\nu(1) - (3 - 2\nu)J_{\nu+1}(1) = 0$ on $(0, \infty)$.

Now, let us consider the function $w_{\alpha,\nu} : \mathbb{D} \to \mathbb{C}$, defined by
\[
w_{\alpha,\nu}(z) = \frac{2\nu}{\alpha z^{\nu/2}} \Gamma(\nu + 1) \left((\alpha - \nu)J_\nu(\sqrt{z}) + \sqrt{z}J'_\nu(\sqrt{z})\right) = \sum_{n \geq 0} \frac{(-1)^n(2n + \alpha)\Gamma(\nu + 1)z^{n+1}}{\alpha \cdot 4^n n!\Gamma(n + \nu + 1)}.
\]

The following sharp result is a common generalization of the above last two Theorems.

Theorem (Baricz, Deniz, Yağmur)

Let $\nu > -\frac{3}{4}$ and $\alpha \geq \frac{2}{4\nu + 3}$. The function $w_{\alpha,\nu}$ is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha$, where $\nu_\alpha$ is the unique root of the equation $(2\alpha - 1)J_\nu(1) - (\alpha - 2\nu + 2)J_{\nu+1}(1) = 0$ on $(-\frac{3}{4}, \infty)$.
We note that when $a = 0$, $b = 1$ and $c = 2 - \nu$, then the fourth Theorem reduces to the fifth Theorem. In this case $\bar{\nu} = 0$, $\nu^\circ$ becomes $\nu_*$ and the inequalities (2) become $4 > 0$, and $4\nu + 3 > 1$. These inequalities give $\nu > -\frac{1}{2}$ and $\nu > 0$, which are not certainly satisfied for $\nu > \nu_*$. 

However, the seventh Theorem is fully covered by the fifth Theorem since when $a = 0$, $b = 1$ and $c = 1 - \nu$, then $\nu = 0$, $\nu^\circ$ becomes $\nu$ and the inequalities (2) become $3 > 0$, and $4\nu + 3 > 2$. These inequalities give $\nu > -\frac{1}{4}$ and $\nu > 0$, which are certainly satisfied for $\nu > \nu_*$.

Finally, we note that when $a = 0$, $b = 1$ and $c = \alpha - \nu$, then the fourth Theorem reduces to the eighth Theorem, but with the additional condition that $\nu > 0$. In this case $\nu = 0$, $\nu^\circ$ becomes $\nu^\alpha$ and the inequalities (2) become $\alpha + 2 > 0$, and $(4\nu + 3)^\alpha > 2$. This means that the fourth Theorem does not cover fully the eighth Theorem.
We note that when \( a = 0, \ b = 1 \) and \( c = 2 - \nu \), then the fourth Theorem reduces to the fifth Theorem. In this case \( \bar{\nu} = 0 \), \( \nu^\circ \) becomes \( \nu_* \) and the inequalities (2) become \( 4 > 0 \), and \( 4\nu + 3 > 1 \). These inequalities give \( \nu > -\frac{1}{2} \) and \( \nu > 0 \), which are not certainly satisfied for \( \nu > \nu_* \). This means that the fourth Theorem does not cover the fifth Theorem.

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We note that when $a = 0$, $b = 1$ and $c = 2 - \nu$, then the fourth Theorem reduces to the fifth Theorem. In this case $\nu = 0$, $\nu^\circ$ becomes $\nu_\star$ and the inequalities (2) become $4 > 0$, and $4\nu + 3 > 1$. These inequalities give $\nu > -\frac{1}{2}$ and $\nu > 0$, which are not certainly satisfied for $\nu > \nu_\star$. This means that the fourth Theorem does not cover the fifth Theorem.

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Finally, we note that when $a = 0$, $b = 1$ and $c = \alpha - \nu$, then the fourth Theorem reduces to the eighth Theorem, but with the additional condition that $\nu > 0$. In this case $\nu = 0$, $\nu^\circ$ becomes $\nu_\alpha$ and the inequalities (2) become $\alpha + 2 > 0$, and $(4\nu + 3)\alpha > 2$. This means that the fourth Theorem does not cover fully the eighth Theorem.
Theorem (Alkharsani, Baricz, Pogány)

The function

\[ z \mapsto 2^{2\nu} z^{-\frac{\nu}{2} + \frac{3}{4}} \Gamma(\nu + 1)\Gamma(\nu + 2) \left( J_{\nu+1}(\sqrt[4]{z}) I_{\nu} \left( \sqrt[4]{z} \right) + J_{\nu} \left( \sqrt[4]{z} \right) I_{\nu+1} \left( \sqrt[4]{z} \right) \right) \]

is starlike in \( \mathbb{D} \) and all of its derivatives are close-to-convex (and hence univalent) there if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx -0.9427 \ldots \) is the unique root of the next equation on \((-1, \infty)\)

\[(\nu - 1) J_\nu(1) I_{\nu+1}(1) + (\nu - 1) J_{\nu+1}(1) I_\nu(1) = J_\nu(1) I_\nu(1).\]
Theorem (Alkharsani, Baricz, Pogány)

The function
\[ z \mapsto 2^{2\nu} z^{-\frac{\nu}{2} + \frac{3}{4}} \Gamma(\nu + 1) \Gamma(\nu + 2) \left( J_{\nu+1}(4\sqrt{z}) I_{\nu}(4\sqrt{z}) + J_{\nu}(4\sqrt{z}) I_{\nu+1}(4\sqrt{z}) \right) \]
is starlike in \( \mathbb{D} \) and all of its derivatives are close-to-convex (and hence univalent) there if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx -0.9427 \ldots \) is the unique root of the next equation on \((-1, \infty)\)

\[ (\nu - 1) J_{\nu}(1) I_{\nu+1}(1) + (\nu - 1) J_{\nu+1}(1) I_{\nu}(1) = J_{\nu}(1) I_{\nu}(1). \]

Theorem (Alkharsani, Baricz, Pogány)

The function
\[ z \mapsto 2^{2\nu} z^{-\frac{\nu}{2} + \frac{1}{2}} \Gamma^2(\nu + 1) J_{\nu}(4\sqrt{z}) I_{\nu}(4\sqrt{z}) \]
is starlike in \( \mathbb{D} \) and all of its derivatives are close-to-convex (and hence univalent) there if and only if \( \nu \geq \nu^* \), where \( \nu^* \approx -0.4336 \ldots \) is the unique root of the next equation on \((-1, \infty)\)

\[ J_{\nu+1}(1) I_{\nu}(1) - J_{\nu}(1) I_{\nu+1}(1) = (\nu + 1) J_{\nu}(1) I_{\nu}(1). \]
Lemma (Alkharsani, Baricz, Pogány)

If \( \nu > -1 \) and \( z \in \mathbb{C} \), then we have the next power series representation

\[
J_{\nu+1}(z)I_{\nu}(z) + J_{\nu}(z)I_{\nu+1}(z) = 2 \sum_{n\geq 0} \frac{(-1)^n \left( \frac{z}{2} \right)^{2\nu+4n+1}}{n!\Gamma(\nu + n + 1)\Gamma(\nu + 2n + 2)}.
\]
Lemma (Alkharsani, Baricz, Pogány)

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\]

Lemma (Alkharsani, Baricz, Pogány)

If \( \nu > -1 \) and \( z \in \mathbb{C} \), then we have the next Hadamard factorization

\[
2^{2\nu} z^{-2\nu-1} \Gamma(\nu + 1)\Gamma(\nu + 2) (J_{\nu+1}(z)I_{\nu}(z) + J_{\nu}(z)I_{\nu+1}(z)) = \prod_{n \geq 1} \left( 1 - \frac{z^4}{\gamma_{\nu,n}^4} \right),
\]

where \( \gamma_{\nu,n} \) is the \( n \)th positive zero of the function \( z \mapsto J_{\nu+1}(z)I_{\nu}(z) + J_{\nu}(z)I_{\nu+1}(z) \). Moreover, the zeros \( \gamma_{\nu,n} \) satisfy the interlacing inequalities \( j_{\nu,n} < \gamma_{\nu,n} < j_{\nu,n+1} \) and \( j_{\nu,n} < \gamma_{\nu,n} < j_{\nu+1,n} \) for \( n \in \mathbb{N} \) and \( \nu > -1 \), where \( j_{\nu,n} \) stands for the \( n \)th positive zero of the Bessel function \( J_{\nu} \).
It is worth to mention that the inequalities $j_{\nu,n} < \gamma_{\nu,n} < j_{\nu+1,n}$ were proved for $\nu \geq -\frac{1}{2}$ by Lorch. To prove the above results we will need also the following preliminary result. Note that this result was proved also earlier by Lorch for the case $\nu \geq -\frac{1}{2}$. Our proof is just a slight modification of the proof made by Lorch.
It is worth to mention that the inequalities \( j_{\nu,n} < \gamma_{\nu,n} < j_{\nu+1,n} \) were proved for \( \nu \geq -\frac{1}{2} \) by Lorch. To prove the above results we will need also the following preliminary result. Note that this result was proved also earlier by Lorch for the case \( \nu \geq -\frac{1}{2} \). Our proof is just a slight modification of the proof made by Lorch.

**Lemma (Alkharsani, Baricz, Pogány)**

*The positive zeros of \( z \mapsto J_{\nu}(z)I'_{\nu}(z) - J'_{\nu}(z)I_{\nu}(z) \) increase with \( \nu \) on \((-1, \infty)\).*


