The new-is-better-than-used class of life distributions

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Abstract. In this talk our aim is to present the new-is-better-than-used (NBU) and increasing-proportionate-failure-rate (IPFR) class of life distributions by using the usual concept of log-convex (log-concave) and geometrically convex (concave) functions. We illustrate the general results via some common univariate distributions, like gamma, Weibull, chi, chi-square. Moreover, we show that the generalized Marcum $Q$-function, which is frequently used in information theory, satisfies the NBU property, while the gamma-gamma distribution has the IPFR property.
1. Log-convex (log-concave) distributions
   - Some special class of life distributions
   - Some general results concerning the NBU class of life distributions
   - Application to some common univariate distributions

2. The generalized Marcum $Q$-function
   - The definition of the generalized Marcum $Q$-function
   - Probabilistic interpretation of the generalized Marcum $Q$-function
   - The NBU property of the generalized Marcum $Q$-function

3. The gamma-gamma distribution
   - Definition of the gamma-gamma distribution
   - The IPFR property of the gamma-gamma distribution
Definition

Let $a, b > 0$ such that $a < b$. The function $f : [a, b] \rightarrow (0, \infty)$ is

1. **convex (concave)**, if for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
   \[
   f(\alpha x + (1 - \alpha)y) \leq (\geq) \alpha f(x) + (1 - \alpha)f(y);
   \]

2. **log-convex (log-concave)**, if the natural logarithm of $f$ is convex, that is, for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
   \[
   f(\alpha x + (1 - \alpha)y) \leq (\geq) [f(x)]^\alpha [f(y)]^{1-\alpha};
   \]

3. **geometrically convex (geometrically concave)**, if the natural logarithm of $f$ is convex in $\ln x$, that is, for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
   \[
   f(x^\alpha y^{1-\alpha}) \leq (\geq) [f(x)]^\alpha [f(y)]^{1-\alpha}.
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Definition of convex (concave) functions

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If $f$ is differentiable, then the following affirmations are valid:

1. $f$ is convex (concave) $\iff x \mapsto f'(x)$ increasing (decreasing)

2. $f$ is log-convex (log-concave) $\iff x \mapsto f'(x)/f(x)$ increasing (decreasing)

3. $f$ is geometrically convex (geometrically concave) $\iff x \mapsto xf'(x)/f(x)$ increasing (decreasing)

Let $f : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ be a continuously differentiable probability distribution function (pdf). Further, let $F, \overline{F} : [a, b] \to [0, 1]$, defined by

$$F(x) = \int_a^x f(t) \, dt \quad \text{and} \quad \overline{F}(x) = 1 - F(x) = \int_x^b f(t) \, dt,$$

be the corresponding cumulative distribution function (cdf) and survival (or reliability) function. Moreover, let $F_l, \overline{F}_r : [a, b] \to (0, \infty)$, defined by

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be the left-hand integral of the cdf and the right-hand integral of the complementary cumulative distribution function (ccdf).
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The following implications are true:

1. \( f \) is log-concave \( \Rightarrow \) \( F \) is log-concave \( \Rightarrow \) \( F_l \) is log-concave.

2. \( f(a) = 0 \) and \( f \) is log-convex \( \Rightarrow \) \( F \) is log-convex \( \Rightarrow \) \( F_l \) is log-convex.

3. \( f \) is log-concave \( \Rightarrow \) \( \overline{F} \) is log-concave \( \Rightarrow \) \( \overline{F}_r \) is log-concave.

4. \( f(b) = 0 \) and \( f \) is log-convex \( \Rightarrow \) \( \overline{F} \) is log-convex \( \Rightarrow \) \( \overline{F}_r \) is log-convex.

Here, and throughout in the sequel, \( f(a) \) and \( f(b) \) should be understood as \( f(a^+) = \lim_{x \searrow a} f(x) \) and \( f(b^-) = \lim_{x \nearrow b} f(x) \) if the corresponding pdf is not defined in \( a \) or \( b \).

see

Log-concave distributions

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Log-concave distributions

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Theorem

Let \([a, b] \subseteq (0, \infty)\), and let \(f : [a, b] \rightarrow (0, \infty)\), \(F : [a, b] \rightarrow [0, 1]\), \(\overline{F} : [a, b] \rightarrow [0, 1]\) be the probability distribution function, cumulative distribution function and survival function (complementary cumulative distribution function) of an absolutely continuous distribution. The following assertions are true:

1. If \(f\) is geometrically concave, then \(F\) and \(\overline{F}\) are geometrically concave too.
2. If \(af(a) = 0\) and \(f\) is geometrically convex, then \(F\) is geometrically convex too.
3. If \(bf(b) = 0\) and \(f\) is geometrically convex, then \(\overline{F}\) is also geometrically convex.

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Some special class of life distributions

1. **IFR** (increasing failure rate): an absolutely continuous distribution (with support \((0, \infty)\)) is in the class IFR if the hazard function \(x \mapsto R(x) = -\ln F(x)\) is convex, that is, the failure rate

\[
x \mapsto r(x) = R'(x) = f(x)/F(x) = -F'(x)/F(x)
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is increasing on \((0, \infty)\). Equivalently, an absolutely continuous distribution (with support \((0, \infty)\)) is in the class IFR if for all \(y > 0\) the function \(x \mapsto F(x + y)/F(x)\) is decreasing on \((0, \infty)\). The class **DFR** (decreasing failure rate) is defined analogously: such kind of distributions for which \(x \mapsto F(x + y)/F(x) = P(X \geq x + y|X \geq x)\) is decreasing for any \(y > 0\).

2. **NBU** (new is better than used): a distribution is in this class if for each \(x, y > 0\) we have

\[
F(x + y) \leq F(x)F(y).
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Since the above inequality can be rewritten as

\[
F(x + y)/F(x) = P(X \geq x + y|X \geq x) \leq P(X \geq y) = F(y),
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in the case of a NBU distribution the chances (to survive) of a newborn individual are better than of another individual who lived already some time. It is important to note that a distribution is IFR if and only if its survival function is log-concave. Moreover, it can be shown that IFR \(\subset\) NBU.
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\[ x \mapsto r(x) = R'(x) = f(x)/F(x) = -F'(x)/F(x) \]

is increasing on \((0, \infty)\). Equivalently, an absolutely continuous distribution (with support \((0, \infty)\)) is in the class IFR if for all \(y > 0\) the function \(x \mapsto F(x + y)/F(x)\) is decreasing on \((0, \infty)\). The class **DFR** (decreasing failure rate) is defined analogously: such kind of distributions for which \(x \mapsto F(x + y)/F(x) = P(X \geq x + y|X \geq x)\) is decreasing for any \(y > 0\).

2. **NBU** (new is better than used): a distribution is in this class if for each \(x, y > 0\) we have \(F(x + y) \leq F(x)F(y)\). Since the above inequality can be rewritten as

\[ F(x + y)/F(x) = P(X \geq x + y|X \geq x) \leq P(X \geq y) = F(y), \]

in the case of a **NBU** distribution the chances (to survive) of a newborn individual are better than of another individual who lived already some time. It is important to note that a distribution is **IFR** if and only if its survival function is log-concave. Moreover, it can be shown that **IFR** \(\subset\) **NBU**.
Suppose that a cancer specialist believes that a patient newly diagnosed as having stage 1 neck cancer has a smaller chance of survival than does a patient who, following a similar initial diagnosis, has received the treatment (say a radiation implant) and then has survived 5 years. How can such a claim be tested?


IPFR (increasing proportionate failure rate): a distribution is in this class if the function

\[ x \mapsto xr(x) = xf(x)/\overline{F}(x) = -x\overline{F}'(x)/\overline{F}(x) \]

is increasing on \((0, \infty)\). Observe that this condition means exactly that \(\overline{F}\) is geometrically concave, of which sufficient condition is that \(f\) is geometrically concave, according to the previous theorem. Note that the expression \(x \mapsto 1 + xf'(x)/f(x)\) is called ISE (income-share elasticity) in the literature. It can be shown that \(\text{IFR} \subset \text{IPFR}\).

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Log-convex (log-concave) distributions

Some general results concerning the NBU class of life distributions

Lemma

Let us consider the continuously differentiable function $\varphi : [0, \infty) \rightarrow (0, \infty)$. If $\varphi(0) \geq 1$ and $\varphi$ is log-concave, then for all $x, y \geq 0$ we have $\varphi(x + y) \leq \varphi(x)\varphi(y)$. Moreover, if $\varphi(0) \leq 1$ and $\varphi$ is log-convex, then the above inequality is reversed.

Theorem

Let $f$ be a continuously differentiable density function which has support $[0, \infty)$. If $f$ is log-concave, then for all $x, y \geq 0$ we have

$$F(x + y) \leq F(x)F(y).$$

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where \( u : [0, \infty) \to (0, \infty) \) is a continuously differentiable function such that \( t \mapsto e^{-t} u(t) \) is integrable. Clearly we have that \( \log f(x)'' = \log u(x)'' \). Consider the survival function \( F : [0, \infty) \to (0, 1] \), defined by

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Then clearly \( F(0) = 1 \) and

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F(x) = \int_x^\infty e^{-t} u(t) \, dt \div \int_0^\infty e^{-t} u(t) \, dt.
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Consider the following distributions: Weibull distribution, chi-square distribution and chi distribution. These distributions have support $[0, \infty)$ and have density functions for $p > 0$ as follows:

$$f_2(x) = px^{p-1}e^{-x^p}, \quad f_3(x) = \frac{x^{(p-2)/2}e^{-x/2}}{2^{p/2}\Gamma(p/2)}$$

and

$$f_4(x) = \frac{x^{p-1}e^{-x^2/2}}{2^{(p-2)/2}\Gamma(p/2)}.$$

Recall that the Weibull distribution with $p = 2$ – as well as the chi distribution with $p = 2$ – is sometimes known as Rayleigh distribution and the chi distribution with $p = 3$ is sometimes called the Maxwell distribution. With some computations we get

$$[\log f_2(x)]'' = \frac{1 - p}{x^2} (1 + px^p), \quad [\log f_3(x)]'' = \frac{2 - p}{2x^2}$$

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Application to some common univariate distributions

Thus the density function $f_2$ of the Weibull distribution is log-concave if $p \geq 1$ and is log-convex if $p \in (0, 1]$. Moreover, it is easy to verify that if $p \in (0, 1]$, then $f_2(\infty) = 0$. Analogously, the density function $f_3$ of the chi-square distribution is log-concave if $p \geq 2$, is log-convex if $p \in (0, 2]$ and $f_3(\infty) = 0$. Finally, note that the density function $f_4$ of the chi distribution is log-concave too when $p \geq 1$. Now, let us define the survival functions of these distributions $F_i : [0, \infty) \to (0, 1]$ by

$$F_i(x) = \int_x^\infty f_i(t) \, dt,$$

where $i = 2, 3, 4$. Clearly we have $F_i(0) = 1$ for each $i = 2, 3, 4$.

**Corollary**

If $p \geq 1$ then for all $x, y \geq 0$ we have the inequality $F_i(x + y) \leq F_i(x)F_i(y)$, where $i = 2, 4$. When $p \in (0, 1]$ and $i = 2$ the above inequality is reversed. If $p \geq 2$ then for all $x, y \geq 0$ the inequality $F_3(x + y) \leq F_3(x)F_3(y)$ holds. Moreover, when $p \in (0, 2]$ the above inequality is reversed.

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Now for all $p > 0$ and $x \in \mathbb{R}$ let

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\Gamma(p, x) = \int_x^\infty t^{p-1} e^{-t} \, dt, \quad \gamma(p, x) = \int_0^x t^{p-1} e^{-t} \, dt
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\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} \, dt
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denote the upper incomplete gamma function, the lower incomplete gamma function and the gamma function, respectively. Recently, Ismail and Laforgia with clever use of Rolle’s theorem proved that the function $q : [0, \infty) \to (0, 1]$, defined by $q(x) := \Gamma(p, x)/\Gamma(p)$, when $p \geq 1$ satisfies the following inequality

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Moreover, they showed that when $p \in (0, 1]$ the above inequality is reversed. Observe that the inequality (2) is actually the new-is-better-than-used property for the gamma distribution, that is, the gamma distribution belongs to the NBU class.

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Application to some common univariate distributions

A function \( f \) with domain \((0, \infty)\) is said to be completely monotonic if it possesses derivatives \( f^{(n)} \) for all \( n = 1, 2, 3, \ldots \) and if \((-1)^n f^{(n)}(x) \geq 0\) for all \( x > 0 \). Due to Kimberling we know that if the continuous function \( h : [0, \infty) \rightarrow (0, 1] \) is completely monotonic on \((0, \infty)\), then we get that \( x \mapsto \log h(x) \) is super-additive, i.e. for all \( x, y \geq 0 \) we have \( h(x)h(y) \leq h(x+y) \). We note that the reverse of (2) is actually an immediate consequence of Kimberling’s result. To prove this, first let us consider \( p = 1 \). Then \( q(x) = e^{-x} \) and clearly we have equality in (2). Now suppose that \( p \in (0, 1) \). Then from the Leibniz rule for derivatives we have

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(-1)^n q^{(n)}(x) \Gamma(p) = (-1)^n \frac{\partial^n \Gamma(p, x)}{\partial x^n} = (-1)^n \frac{\partial^{n-1} [x^{p-1}(-e^{-x})]}{\partial x^{n-1}}
\]

\[
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A function $f$ with domain $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $f^{(n)}$ for all $n = 1, 2, 3, \ldots$ and if $(-1)^nf^{(n)}(x) \geq 0$ for all $x > 0$. Due to Kimberling we know that if the continuous function $h : [0, \infty) \to (0, 1]$ is completely monotonic on $(0, \infty)$, then we get that $x \mapsto \log h(x)$ is super-additive, i.e. for all $x, y \geq 0$ we have $h(x)h(y) \leq h(x + y)$. We note that the reverse of (2) is actually an immediate consequence of Kimberling’s result. To prove this, first let us consider $p = 1$. Then $q(x) = e^{-x}$ and clearly we have equality in (2). Now suppose that $p \in (0, 1)$. Then from the Leibniz rule for derivatives we have

$$(-1)^nq^{(n)}(x)\Gamma(p) = (-1)^n\frac{\partial^n\Gamma(p, x)}{\partial x^n} = (-1)^n\frac{\partial^{n-1}[x^{p-1}(-e^{-x})]}{\partial x^{n-1}}$$

$$= e^{-x}\sum_{k=0}^{n-1}C^n_{n-1} \prod_{m=1}^{k}(m-p)x^{p-k-1} \geq 0$$

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Application to some common univariate distributions

A function $f$ with domain $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $f^{(n)}$ for all $n = 1, 2, 3, \ldots$ and if $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$. Due to Kimberling we know that if the continuous function $h : [0, \infty) \to (0, 1]$ is completely monotonic on $(0, \infty)$, then we get that $x \mapsto \log h(x)$ is super-additive, i.e. for all $x, y \geq 0$ we have $h(x)h(y) \leq h(x + y)$. We note that the reverse of (2) is actually an immediate consequence of Kimberling’s result. To prove this, first let us consider $p = 1$. Then $q(x) = e^{-x}$ and clearly we have equality in (2). Now suppose that $p \in (0, 1)$. Then from the Leibniz rule for derivatives we have

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see


Now, consider again the density function $f : [0, \infty) \rightarrow (0, \infty)$, defined by

$$f(x) = \frac{e^{-x} u(x)}{\int_0^\infty e^{-t} u(t) \, dt},$$

where $u : [0, \infty) \rightarrow (0, \infty)$ is a continuously differentiable function such that $t \mapsto e^{-t} u(t)$ is integrable. Then for

$$\overline{F}(x) = \int_x^\infty e^{-t} u(t) \, dt / \int_0^\infty e^{-t} u(t) \, dt,$$

we have the following result based on Kimberling’s result.

**Corollary**

If the function $u$ is completely monotonic, then $\overline{F}$ satisfies the inequality

$$\overline{F}(x)\overline{F}(y) \leq \overline{F}(x + y)$$

for all $x, y \geq 0$.

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see

The generalized Marcum $Q$-function

Let $l_\nu$ be the modified Bessel function of the first kind of order $\nu$

$$l_\nu(x) = \sum_{k \geq 0} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu + k + 1)},$$

and let $Q_\nu(a, b)$ be the generalized Marcum $Q$–function, defined by

$$Q_\nu(a, b) = \begin{cases} \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} l_{\nu-1}(at) \, dt, & \text{if } a > 0 \\ \frac{1}{2^{\nu-1}\Gamma(\nu)} \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \, dt, & \text{if } a = 0 \end{cases},$$

where $b \geq 0$ and $\nu > 0$. Notice that for each $\nu$, $a > 0$ we have

$$Q_\nu(a, 0) = \frac{1}{a^{\nu-1}} \int_0^\infty t^\nu e^{-\frac{t^2+a^2}{2}} l_{\nu-1}(at) \, dt = 1,$$

and when $a = 0$ clearly we have for each $\nu > 0$ that

$$Q_\nu(0, 0) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \int_0^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \, dt = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-u} u^{\nu-1} \, du = 1.$$

Thus in fact for all $b \geq 0$ and $\nu > 0$ we have

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The generalized Marcum $Q$-function

Let $I_\nu$ be the modified Bessel function of the first kind of order $\nu$

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Let $X_1, X_2, \ldots, X_n$ be random variables that are normally distributed with unit variance and nonzero mean $\mu_1, \mu_2, \ldots, \mu_n$, then the random variable $[X_1^2 + X_2^2 + \ldots + X_n^2]^{1/2}$ has the non-central chi distribution with $n = 1, 2, 3, \ldots$ degrees of freedom and non-centrality parameter $\tau = [\mu_1^2 + \mu_2^2 + \ldots + \mu_n^2]^{1/2}$. The pdf $\chi_{n,\tau} : (0, \infty) \to (0, \infty)$ of the non-central chi distribution is defined as

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\chi_{n,\tau}(x) = 2^{-n/2} e^{-x^2/2} \sum_{k \geq 0} \frac{x^{n+2k-1}(\tau/2)^{2k}}{\Gamma(n/2 + k) k!} = \tau e^{-x^2/2} \left( \frac{x}{\tau} \right)^{n/2} I_{n/2-1}(\tau x).
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Observe that when $\mu_1 = \mu_2 = \ldots = \mu_n = 0$, i.e. $\tau = 0$, the above distribution reduces to the classical chi distribution with pdf $\chi_{n,0} : (0, \infty) \to (0, \infty)$ given by

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Thus taking into account the above definitions and (4), in particular, when $n = 2\nu$ is an integer the generalized Marcum $Q-$function is exactly the reliability (or survival) function of the non-central chi distribution with $2\nu$ degrees of freedom and non-centrality parameter $\tau = a$. 

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Let $Y_1, Y_2, \ldots, Y_m$ be random variables that are normally distributed with unit variance and nonzero mean $\gamma_i$, where $i = 1, 2, \ldots, m$. It is known that $Y_1^2 + Y_2^2 + \ldots + Y_m^2$ has the non-central chi-square distribution with $m = 1, 2, 3, \ldots$ degrees of freedom and non-centrality parameter $\lambda = \gamma_1^2 + \gamma_2^2 + \ldots + \gamma_m^2$. The pdf $\chi^2_{m, \lambda} : (0, \infty) \rightarrow (0, \infty)$ of the non-central chi-square distribution is defined as

$$\chi^2_{m, \lambda}(x) = 2^{-m/2} e^{-(x+\lambda)/2} \sum_{k \geq 0} \frac{x^{m/2+k-1} (\lambda/4)^k}{\Gamma(m/2+k) k!} = \frac{e^{-(x+\lambda)/2}}{2} \left( \frac{x}{\lambda} \right)^{m/2-1/2} I_{m/2-1}(\sqrt{\lambda}x).$$

Recall that when $\gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$, i.e. $\lambda = 0$, the above distribution reduces to the classical chi-square distribution. The pdf $\chi^2_{m, 0} : (0, \infty) \rightarrow (0, \infty)$ of this distribution is given by

$$\chi^2_m(x) = \chi^2_{m, 0}(x) = \frac{x^{m/2-1} e^{-x/2}}{2^{m/2} \Gamma(m/2)}.$$

Now from (3) and (4) it is easy to verify that

$$Q_{\nu} \left( \sqrt{a}, \sqrt{b} \right) = \begin{cases} 
1 - \frac{1}{2} \int_0^b \left( \frac{t}{a} \right)^{\nu-1/2} e^{-t+a} I_{\nu-1} \left( \sqrt{at} \right) dt, & \text{if } a > 0 \\
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Let $Y_1, Y_2, \ldots, Y_m$ be random variables that are normally distributed with unit variance and nonzero mean $\gamma_i$, where $i = 1, 2, \ldots, m$. It is known that $Y_1^2 + Y_2^2 + \ldots + Y_m^2$ has the non-central chi-square distribution with $m = 1, 2, 3, \ldots$ degrees of freedom and non-centrality parameter $\lambda = \gamma_1^2 + \gamma_2^2 + \ldots + \gamma_m^2$. The pdf $\chi^2_{m, \lambda} : (0, \infty) \to (0, \infty)$ of the non-central chi-square distribution is defined as
\[
\chi^2_{m, \lambda}(x) = 2^{-m/2} e^{-(x+\lambda)/2} \sum_{k \geq 0} \frac{x^{m/2+k-1} (\lambda/4)^k}{\Gamma(m/2 + k) k!} = \frac{e^{-(x+\lambda)/2}}{2} \left( \frac{x}{\lambda} \right)^{m/4-1/2} I_{m/2-1}(\sqrt{\lambda}x).
\]
Recall that when $\gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$, i.e. $\lambda = 0$, the above distribution reduces to the classical chi-square distribution. The pdf $\chi^2_{m, 0} : (0, \infty) \to (0, \infty)$ of this distribution is given by
\[
\chi^2_{m}(x) = \chi^2_{m, 0}(x) = \frac{x^{m/2-1} e^{-x/2}}{2^{m/2} \Gamma(m/2)}.
\]
Now from (3) and (4) it is easy to verify that
\[
Q_{\nu} \left( \sqrt{a}, \sqrt{b} \right) = \begin{cases} 
1 - \frac{1}{2} \int_0^b \left( \frac{t}{a} \right)^{\nu - \frac{1}{2}} e^{-\frac{t+a}{2}} I_{\nu-1} \left( \sqrt{at} \right) \, dt, & \text{if } a > 0 \\
1 - \frac{1}{2^\nu \Gamma(\nu)} \int_0^b t^{\nu-1} e^{-\frac{t}{2}} \, dt, & \text{if } a = 0
\end{cases}, \tag{5}
\]
i.e. the function $Q_{\nu} \left( \sqrt{a}, \sqrt{b} \right)$ in particular is the survival function of the non-central chi-square distribution with $m = 2\nu$ degrees of freedom and non-centrality parameter $\lambda = a$. 

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$$\chi^2_{m,\lambda}(x) = 2^{-m/2}e^{-(x+\lambda)/2} \sum_{k \geq 0} \frac{x^{m/2+k-1}(\lambda/4)^k}{\Gamma(m/2 + k)k!} = \frac{e^{-(x+\lambda)/2}}{2} \left( \frac{x}{\lambda} \right)^{m/4-1/2} I_{m/2-1}(\sqrt{\lambda x}).$$

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i.e. the function $Q_{\nu}(\sqrt{a}, \sqrt{b})$ in particular is the survival function of the non-central chi-square distribution with $m = 2\nu$ degrees of freedom and non-centrality parameter $\lambda = a$. 
The NBU property of the generalized Marcum $Q$-function

**Theorem**

Let $a \geq 0$ and $\nu > 1$. Then the following assertions are true:

1. the function $b \mapsto Q_\nu(a, \sqrt{b})$ is strictly log-concave on $(0, \infty)$;
2. the function $b \mapsto Q_\nu(a, b)$ is strictly log-concave on $(0, \infty)$;
3. the inequalities

\[ Q_\nu \left( a, \sqrt{b_1 + b_2} \right) < Q_\nu(a, \sqrt{b_1}) Q_\nu(a, \sqrt{b_2}) < Q_\nu^2 \left( a, \sqrt{\frac{b_1 + b_2}{2}} \right), \]

hold true for all $b_1, b_2 > 0$ and $b_1 \neq b_2$. Moreover, for all $b_1, b_2 > 0$ and $b_1 \neq b_2$ we have

\[ Q_\nu(a, b_1 + b_2) < Q_\nu \left( a, \sqrt{b_1^2 + b_2^2} \right) < Q_\nu(a, b_1) Q_\nu(a, b_2) \]

\[ < Q_\nu^2 \left( a, \sqrt{\frac{b_1^2 + b_2^2}{2}} \right) < Q_\nu^2 \left( a, \frac{b_1 + b_2}{2} \right). \]

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see

The definition of the gamma-gamma distribution

The probability density function \( f_{a,b,\alpha} : (0, \infty) \rightarrow (0, \infty) \) of the three parameter gamma-gamma random variable is defined by

\[
f_{a,b,\alpha}(u) = \frac{2(ab)^{a+b/2}u^{a+b/2-1}}{\Gamma(a)\Gamma(b)\alpha^{a+b/2}} K_{a-b}\left(2\sqrt{\frac{ab}{\alpha}} u\right),
\]

where \( a, b > 0 \) are the distribution shaping parameters, \( K_{\nu} \), defined by

\[
K_{\nu}(u) = \int_{0}^{\infty} e^{-u\cosh t} \cosh(\nu t) \, dt,
\]

stands for the modified Bessel function of the second kind, and \( \alpha > 0 \) is the mean of the gamma-gamma random variable. The gamma-gamma distribution is produced from the product of two independent gamma random variables and has been widely used in a variety of applications, for example in modeling various types of land and sea radar clutters, in modeling the effects of the combined fading and shadowing phenomena, encountered in the mobile communications channels. Of particular interest is the application of the gamma-gamma distribution in optical wireless systems, where transmission of optical signals through the atmosphere is involved.

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The probability density function $f_{a,b,\alpha} : (0, \infty) \to (0, \infty)$ of the three parameter gamma-gamma random variable is defined by

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where $a, b > 0$ are the distribution shaping parameters, $K_{\nu}$, defined by

$$K_{\nu}(u) = \int_0^\infty e^{-ucosh t} cosh(\nu t) dt,$$

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see


The gamma-gamma distribution

Now, consider the functions \( \tilde{f}_{a, b, \alpha} : (0, \infty) \rightarrow (0, \infty) \) and \( F_{a, b, \alpha} : (0, \infty) \rightarrow (0, 1) \) defined by

\[
\tilde{f}_{a, b, \alpha}(u) = f_{a, b, \alpha} \left( \frac{\alpha u^2}{4ab} \right) = \frac{2^{3-(a+b)}(ab)u^{a+b-2}}{\alpha \Gamma(a)\Gamma(b)} K_{a-b}(u)
\]

and

\[
F_{a, b, \alpha}(u) = \int_0^u f_{a, b, \alpha}(t) \, dt = \frac{1}{\Gamma(a)\Gamma(b)} \cdot G^{2, 1}_{1, 3} \left[ \frac{ab}{\alpha} \left| \begin{array}{c} 1 \\ a, b, 0 \end{array} \right. \right],
\]

where \( G^{1, 2}_{1, 3} \) is a Meijer \( G \)–function. Here \( \tilde{f}_{a, b, \alpha} \) is just a transformation of the pdf \( f_{a, b, \alpha} \), while \( F_{a, b, \alpha} \) is the cumulative distribution function of the gamma-gamma distribution.

**Theorem**

Let \( a, b, \alpha > 0 \). Then the following assertions are true:

1. \( u \mapsto \frac{\tilde{f}_{a, b, \alpha}'(u)}{\tilde{f}_{a, b, \alpha}(u)} \) is strictly decreasing on \( (0, \infty) \);
2. \( u \mapsto \frac{uf_{a, b, \alpha}'(u)}{f_{a, b, \alpha}(u)} \) is strictly decreasing on \( (0, \infty) \);
3. \( u \mapsto \frac{uf_{a, b, \alpha}'(u)}{f_{a, b, \alpha}(u)} \) is strictly decreasing on \( (0, \infty) \);
4. \( u \mapsto \frac{F_{a, b, \alpha}'(u)}{F_{a, b, \alpha}(u)} \) is strictly decreasing on \( (0, \infty) \).

see

The IPFR property of the gamma-gamma distribution

Now, consider the functions $\tilde{f}_{a,b,\alpha} : (0, \infty) \to (0, \infty)$ and $F_{a,b,\alpha} : (0, \infty) \to (0, 1)$ defined by

$$\tilde{f}_{a,b,\alpha}(u) = f_{a,b,\alpha} \left( \frac{\alpha u^2}{4ab} \right) = \frac{2^{3-(a+b)}(ab)u^{a+b-2}}{\alpha \Gamma(a)\Gamma(b)} K_{a-b}(u)$$

and

$$F_{a,b,\alpha}(u) = \int_0^u f_{a,b,\alpha}(t) \, dt = \frac{1}{\Gamma(a)\Gamma(b)} \cdot \left. G_{1,3}^{2,1} \left[ \begin{array}{c} ab \\ \alpha \end{array} | a, b, 0 \right] \right|_0^1,$$

where $G_{1,3}^{2,1}$ is a Meijer $G$–function. Here $\tilde{f}_{a,b,\alpha}$ is just a transformation of the pdf $f_{a,b,\alpha}$, while $F_{a,b,\alpha}$ is the cumulative distribution function of the gamma-gamma distribution.

**Theorem**

Let $a, b, \alpha > 0$. Then the following assertions are true:

1. $u \mapsto \frac{u \tilde{f}'_{a,b,\alpha}(u)}{\tilde{f}_{a,b,\alpha}(u)}$ is strictly decreasing on $(0, \infty)$;
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and

$$
F_{a,b,\alpha}(u) = \int_0^u f_{a,b,\alpha}(t) \, dt = \frac{1}{\Gamma(a)\Gamma(b)} \cdot G_{1,3}^{2,1} \left[ \frac{ab}{\alpha} \right| \begin{array}{c} 1 \\ a, b, 0 \end{array} \right],
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where $G_{1,3}^{2,1}$ is a Meijer $G-$function. Here $\tilde{f}_{a,b,\alpha}$ is just a transformation of the pdf $f_{a,b,\alpha}$, while $F_{a,b,\alpha}$ is the cumulative distribution function of the gamma-gamma distribution.

Theorem

Let $a, b, \alpha > 0$. Then the following assertions are true:

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Theorem

Let $a, b, \alpha > 0$. Then the following assertions are true:

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Now, consider the functions \( \tilde{f}_{a,b,\alpha} : (0, \infty) \to (0, \infty) \) and \( F_{a,b,\alpha} : (0, \infty) \to (0, 1) \) defined by

\[
\tilde{f}_{a,b,\alpha}(u) = f_{a,b,\alpha} \left( \frac{\alpha u^2}{4ab} \right) = \frac{2^{3-(a+b)}(ab)u^{a+b-2}}{\alpha \Gamma(a) \Gamma(b)} K_{a-b}(u)
\]

and

\[
F_{a,b,\alpha}(u) = \int_0^u f_{a,b,\alpha}(t) \, dt = \frac{1}{\Gamma(a) \Gamma(b)} \cdot G_{1,3}^{2,1} \left[ \begin{array}{c} \frac{ab}{\alpha} u \\ a, b, 0 \end{array} \right],
\]

where \( G_{1,3}^{2,1} \) is a Meijer G–function. Here \( \tilde{f}_{a,b,\alpha} \) is just a transformation of the pdf \( f_{a,b,\alpha} \), while \( F_{a,b,\alpha} \) is the cumulative distribution function of the gamma-gamma distribution.

Theorem

Let \( a, b, \alpha > 0 \). Then the following assertions are true:

1. \( u \mapsto \frac{\tilde{f}_{a,b,\alpha}'}{f_{a,b,\alpha}}(u)/f_{a,b,\alpha}(u) \) is strictly decreasing on \( (0, \infty) \);
2. \( u \mapsto uf'_{a,b,\alpha}(u)/f_{a,b,\alpha}(u) \) is strictly decreasing on \( (0, \infty) \);
3. \( u \mapsto uf'_{a,b,\alpha}(u)/F_{a,b,\alpha}(u) \) is strictly decreasing on \( (0, \infty) \);
4. \( u \mapsto F'_{a,b,\alpha}(u)/F_{a,b,\alpha}(u) \) is strictly decreasing on \( (0, \infty) \).

see

Proof

1. We know that $K_\nu$ is geometrically concave, and thus we have that the function

$$u \mapsto \frac{u f_{a,b,\alpha}'(u)}{f_{a,b,\alpha}(u)} = a + b - 2 + \frac{u K_{a-b}'(u)}{K_{a-b}(u)}$$

is strictly decreasing on $(0, \infty)$ for all $a, b, \alpha > 0$.

2. Observe that the above part of this theorem actually means that the function $\tilde{f}_{a,b,\alpha}$ is strictly geometrically concave, i.e. for all $a, b, \alpha > 0$, $\lambda \in (0, 1)$ and $u_1, u_2 > 0$, $u_1 \neq u_2$ we have

$$\tilde{f}_{a,b,\alpha} \left( u_1^\lambda u_2^{1-\lambda} \right) > \left[ \tilde{f}_{a,b,\alpha}(u_1) \right]^\lambda \left[ \tilde{f}_{a,b,\alpha}(u_2) \right]^{1-\lambda}.$$

Now, changing in the above inequality $u_i$ with $2 \sqrt{abu_i}/\alpha$, where $i \in \{1, 2\}$, we obtain

$$f_{a,b,\alpha} \left( u_1^\lambda u_2^{1-\lambda} \right) > \left[ f_{a,b,\alpha}(u_1) \right]^\lambda \left[ f_{a,b,\alpha}(u_2) \right]^{1-\lambda}$$

for all $a, b, \alpha > 0$, $\lambda \in (0, 1)$ and $u_1, u_2 > 0$, $u_1 \neq u_2$. This means that $f_{a,b,\alpha}$ is strictly geometrically concave and hence $u \mapsto uf_{a,b,\alpha}'(u)/f_{a,b,\alpha}(u)$ is strictly decreasing on $(0, \infty)$.

3. This follows from the above part of this theorem.

4. The above part of this theorem states that the cumulative distribution function $F_{a,b,\alpha}$ is strictly geometrically concave. Now, by using the fact that $F_{a,b,\alpha}$, as a distribution function, is increasing, for all $a, b, \alpha > 0$, $\lambda \in (0, 1)$ and $u_1, u_2 > 0$, $u_1 \neq u_2$ we have

$$F_{a,b,\alpha} (\lambda u_1 + (1-\lambda)u_2) > F_{a,b,\alpha} \left( u_1^\lambda u_2^{1-\lambda} \right) > \left[ F_{a,b,\alpha}(u_1) \right]^\lambda \left[ F_{a,b,\alpha}(u_2) \right]^{1-\lambda},$$

that is, $F_{a,b,\alpha}$ is strictly log-concave on $(0, \infty)$. 

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Proof

1. We know that $K_\nu$ is geometrically concave, and thus we have that the function

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for all $a, b, \alpha > 0$, $\lambda \in (0, 1)$ and $u_1, u_2 > 0$, $u_1 \neq u_2$. This means that $f_{a,b,\alpha}$ is strictly geometrically concave and hence $u \mapsto uf'_{a,b,\alpha}(u)/f_{a,b,\alpha}(u)$ is strictly decreasing on $(0, \infty)$.

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Proof

1. We know that $K_\nu$ is geometrically concave, and thus we have that the function

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is strictly decreasing on $(0, \infty)$ for all $a, b, \alpha > 0$.

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for all $a, b, \alpha > 0$, $\lambda \in (0, 1)$ and $u_1, u_2 > 0$, $u_1 \neq u_2$. This means that $f_{a,b,\alpha}$ is strictly geometrically concave and hence $u \mapsto uf'_{a,b,\alpha}(u)/f_{a,b,\alpha}(u)$ is strictly decreasing on $(0, \infty)$.

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