The monotone form of l’Hospital’s rule and geometrically concave distributions

Árpád Baricz
https://sites.google.com/site/bariczocsi/
e-mail: bariczocsi@yahoo.com

Babeș-Bolyai University, Department of Economics, Romania
Óbuda University, Institute of Applied Mathematics, Hungary

Seminar of Indian Statistical Institute, Chennai Centre, Chennai, India
In this talk our aim is to show that if a probability density function is geometrically concave (convex), then the corresponding cumulative distribution function and the survival function are geometrically concave (convex) too, under some assumptions. The proofs are based on the so-called monotone form of l’Hospital’s rule and permit us to extend our results to the case of the concavity (convexity) with respect to Hölder means. To illustrate the applications of the main results, we discuss in details the geometrical concavity of the probability density function, cumulative distribution function and survival function of some common continuous univariate distributions. Moreover, we present a simple alternative proof to Schweizer’s problem related to the Mulholland’s generalization of Minkowski’s inequality.
1. L'Hospital’s rule

2. The monotone form of l’Hospital’s rule
   - Anderson, Vamanamurthy and Vuorinen’s result (1993)
   - The result of Pinelis (2002)
   - Gromov’s result (1982)
   - Wu and Debnath’s result (2009)

3. Applications
   - Log-convex (log-concave) distributions
   - Bagnoli and Bergstrom’s results (2005)
   - Some special class of distributions

4. Searching for common generalizations
   - The case of power means
   - Geometrical concavity of some common univariate distributions
   - Generalization of Minkowski’s inequality
Theorem (L’Hospital’s rule)

Let \(-\infty \leq a < b \leq \infty\) and \(f, g : (a, b) \to \mathbb{R}\) be differentiable functions such that \(g'(x) \neq 0\) for \(x \in (a, b)\). If

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0
\]

or

\[
\lim_{x \to a} f(x) = \pm \lim_{x \to a} g(x) = \pm \infty
\]

and

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = \ell,
\]

then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \ell.
\]

- see G. L’Hospital, Analyse des Infiniment Petits pour l’Intelligence des Lignes Courbes, 1696.

- Bernoulli rule ???
- Guillaume François Antoine, Marquis de L’Hospital (1661-1704)
- Johann Bernoulli (1667-1748)
Theorem

Let \( a, b \in \mathbb{R} \), where \( a < b \), and \( f, g : [a, b) \to \mathbb{R} \) be differentiable functions such that \( g'(x) \neq 0 \) for \( x \in (a, b) \). If the function \( f'/g' \) is increasing (decreasing) on \((a, b)\), then the function

\[
x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)}
\]

is also increasing (decreasing) on \((a, b)\).


Proof. Without loss of generality assume that \( g'(x) > 0 \) for all \( x \in (a, b) \). By applying the Cauchy mean value theorem for \( x \in (a, b) \) there exists \( y \in (a, x) \) such that

\[
\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)} \leq \frac{f'(x)}{g'(x)}.
\]

On the other hand,

\[
\left( \frac{f(x) - f(a)}{g(x) - g(a)} \right)' = \frac{f'(x)(g(x) - g(a)) - g'(x)(f(x) - f(a))}{((g(x) - g(a))^2},
\]

which implies that the function

\[
x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)}
\]

is indeed increasing (decreasing) on \((a, b)\). QED.
• **Glen Douglas Anderson**  
  (Ph.D., University of Michigan, USA, 1965)

• **Mavina Kayahalli Krishnamurthyrao Vamanamurthy** (1934-2009)  
  (Ph.D., University of Michigan, USA, 1969)

• **Matti Keijo Kustaa Vuorinen**  
  (Ph.D., University of Helsinki, Finland, 1977)  
Theorem

Let $-\infty \leq a < b \leq \infty$ and $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable functions such that $g'(x) \neq 0$ for $x \in (a, b)$. Assume also that $\lim_{x \downarrow a} f(x) = \lim_{x \downarrow a} g(x) = 0$ or $\lim_{x \uparrow b} f(x) = \lim_{x \uparrow b} g(x) = 0$. If the function $f'/g'$ is increasing (decreasing) on $(a, b)$, then so is the function $f/g$ on $(a, b)$.

Proof. Without loss of generality assume that $g'(x) > 0$ for $x \in (a, b)$. Assume also that $\lim_{x \downarrow a} f(x) = \lim_{x \downarrow a} g(x) = 0$, and for fixed $x \in (a, b)$ consider the function $h_x : (a, b) \rightarrow \mathbb{R}$,

$$h_x(y) = f'(x)g(y) - g'(x)f(y).$$

The function $h_x$ is differentiable on $(a, b)$, and it is also continuous. On the other hand, for $y \in (a, x)$ we have that

$$\frac{d}{dy} h_x(y) = h'_x(y) = f'(x)g'(y) - g'(x)f'(y) = g'(x)g'(y) \left( \frac{f'(x)}{g'(x)} - \frac{f'(y)}{g'(y)} \right) > (\prec)0.$$

Consequently, the function $h_x$ is increasing (decreasing) on $(a, x)$, and since is continuous, it is increasing (decreasing) on $(a, x]$. However, we have $\lim_{y \downarrow a} h_x(y) = 0$ and thus for each $x \in (a, b)$ we get $h_x(x) > (\prec)0$, which implies that

$$\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)} = \frac{h_x(x)}{g^2(x)} > (\prec)0,$$

that is, the function $f/g$ is also increasing (decreasing) on $(a, b)$. The proof for the case $\lim_{x \uparrow b} f(x) = \lim_{x \uparrow b} g(x) = 0$ is similar. QED.
Iosif Pinelis
(Ph.D., Novisibirsk, Russia, 1982)
http://www.math.mtu.edu/~ipinelis/
Theorem

If the functions $f, g : (0, \infty) \rightarrow (0, \infty)$ are integrable and the function $f/g$ is decreasing, then the function

$$x \mapsto \frac{\int_0^x f(t) \, dt}{\int_0^x g(t) \, dt}$$

is also decreasing.


- **Mikhail Leonidovich Gromov** (Ph.D., Leningrad, Russia, 1969)
**Theorem**

Let \( a, b \in \mathbb{R} \), where \( a < b \), and let \( f, g : (a, b) \to \mathbb{R} \) be differentiable functions such that \( g'(x) \neq 0 \) for each \( x \in (a, b) \). If the function \( f'/g' \) is increasing (decreasing) on \((a, b)\) and there exist \( \lim_{x \to a^-} f(x), \lim_{x \to a^-} g(x) \), then the function

\[
\begin{align*}
  x &\mapsto \frac{f(x) - \lim_{x \to a^-} f(x)}{g(x) - \lim_{x \to a^-} g(x)}
\end{align*}
\]

is also increasing (decreasing) on \((a, b)\). Similarly, if the function \( f'/g' \) is increasing (decreasing) on \((a, b)\) and there exist \( \lim_{x \to b^+} f(x), \lim_{x \to b^+} g(x) \), then the function

\[
\begin{align*}
  x &\mapsto \frac{f(x) - \lim_{x \to b^+} f(x)}{g(x) - \lim_{x \to b^+} g(x)}
\end{align*}
\]

is also increasing (decreasing) on \((a, b)\).

---

Shanhe Wu (University of Longyan, Longyan Fujian, China)
Lokenath Debnath (Ph.D., University of Calcutta, India, 1965)
Convex (concave) functions

Definition

Let $a, b > 0$ such that $a < b$. The function $f : [a, b] \to (0, \infty)$ is

- **convex (concave),** if for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
  \[ f(\alpha x + (1 - \alpha)y) \leq (\geq) \alpha f(x) + (1 - \alpha)f(y); \]

- **log-convex (log-concave),** if the natural logarithm of $f$ is convex, that is, for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
  \[ f(\alpha x + (1 - \alpha)y) \leq (\geq) [f(x)]^\alpha [f(y)]^{1-\alpha}; \]

- **geometrically convex (geometrically concave),** if the natural logarithm of $f$ is convex in $\ln x$, that is, for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have
  \[ f(x^\alpha y^{1-\alpha}) \leq (\geq) [f(x)]^\alpha [f(y)]^{1-\alpha}. \]

If $f$ is differentiable, then the following affirmations are valid:

- $f$ is convex (concave) $\iff \ x \mapsto f'(x)$ increasing (decreasing)
- $f$ is log-convex (log-concave) $\iff \ x \mapsto f'(x)/f(x)$ increasing (decreasing)
- $f$ is geometrically convex (geometrically concave) $\iff \ x \mapsto xf'(x)/f(x)$ increasing (decreasing)
Log-concave distributions

Let \( f : [a, b] \subseteq \mathbb{R} \to (0, \infty) \) be a continuously differentiable probability distribution function (pdf). Further, let \( F, \overline{F} : [a, b] \to [0, 1] \), defined by

\[
F(x) = \int_a^x f(t) \, dt \quad \text{and} \quad \overline{F}(x) = 1 - F(x) = \int_x^b f(t) \, dt,
\]

be the corresponding cumulative distribution function (cdf) and survival (or reliability) function. Moreover, let \( F_l, \overline{F}_r : [a, b] \to (0, \infty) \), defined by

\[
F_l(x) = \int_a^x F(t) \, dt = \int_a^x \int_a^t f(s) \, ds \, dt \quad \text{and} \quad \overline{F}_r(x) = \int_x^b \overline{F}(t) \, dt = \int_x^b \int_t^b f(s) \, ds \, dt,
\]

be the left-hand integral of the cdf and the right-hand integral of the complementary cumulative distribution function (ccdf).

\[ \text{Theorem} \]

The following implications are true:

(a) \( f \) is log-concave \( \Rightarrow \) \( F \) is log-concave \( \Rightarrow \) \( F_l \) is log-concave.

(b) \( f(a) = 0 \) and \( f \) is log-convex \( \Rightarrow \) \( F \) is log-convex \( \Rightarrow \) \( F_l \) is log-convex.

(c) \( f \) is log-concave \( \Rightarrow \) \( \overline{F} \) is log-concave \( \Rightarrow \) \( \overline{F}_r \) is log-concave.

(d) \( f(b) = 0 \) and \( f \) is log-convex \( \Rightarrow \) \( \overline{F} \) is log-convex \( \Rightarrow \) \( \overline{F}_r \) is log-convex.

Here, and throughout in the sequel, \( f(a) \) and \( f(b) \) should be understood as \( f(a^+) = \lim_{x \downarrow a} f(x) \) and \( f(b^-) = \lim_{x \uparrow b} f(x) \) if the corresponding pdf is not defined in \( a \) or \( b \).

• Mark Bagnoli (economist) (Ph.D., Princeton University, USA, 1985)
• Ted Bergstrom (economist) (Ph.D., Stanford University, USA, 1967)
Theorem

Let \([a, b] \subseteq (0, \infty)\), and let \(f : [a, b] \rightarrow (0, \infty)\), \(F : [a, b] \rightarrow [0, 1]\), \(\bar{F} : [a, b] \rightarrow [0, 1]\) be the probability distribution function, cumulative distribution function and survival function (complementary cumulative distribution function) of an absolutely continuous distribution. The following assertions are true:

- If \(f\) is geometrically concave, then \(F\) and \(\bar{F}\) are geometrically concave too.
- If \(af(a) = 0\) and \(f\) is geometrically convex, then \(F\) is geometrically convex too.
- If \(bf(b) = 0\) and \(f\) is geometrically convex, then \(\bar{F}\) is also geometrically convex.

see

Proof. We present only the proof of the first implications of the above two theorems. The rest of the proofs are similar. Since $f$ log-concave, it follows that 

$$x \mapsto f'(x)/f(x) = f'(x)/F'(x) = -f'(x)/\overline{F}'(x)$$

is decreasing on $[a, b]$. By using the monotone form of l'Hospital's rule we obtain that the functions 

$$x \mapsto \frac{f(x) - f(a)}{F(x) - F(a)} = \frac{f(x) - f(a)}{F(x)} = \frac{F'(x)}{F(x)} - \frac{f(a)}{F(x)},$$

$$x \mapsto -\frac{f(x) - f(b)}{F(x) - \overline{F}(b)} = -\frac{f(x) - f(b)}{\overline{F}(x)} = \frac{\overline{F}'(x)}{\overline{F}(x)} + \frac{f(b)}{\overline{F}(x)},$$

are also decreasing on $[a, b]$. Since $F$ is increasing and $\overline{F}$ is decreasing, we get that the functions 

$$x \mapsto F'(x)/F(x)$$

and 

$$x \mapsto \overline{F}'(x)/\overline{F}(x)$$

are decreasing on $[a, b]$, that is, $F$ and $\overline{F}$ are log-concave. Similarly, since $f$ is geometrically concave, it follows that $x \mapsto xf'(x)/f(x)$ is decreasing and thus the function 

$$x \mapsto 1 + xf'(x)/f(x) = [xf(x)]'/F'(x) = -[xf(x)]'/\overline{F}'(x)$$

is also decreasing on $[a, b]$. By using again the monotone form of l'Hospital's rule we get that the functions 

$$x \mapsto \frac{xf(x) - af(a)}{F(x) - F(a)} = \frac{xf(x) - af(a)}{F(x)} = \frac{xF'(x)}{F(x)} - \frac{af(a)}{F(x)},$$

$$x \mapsto -\frac{xf(x) - bf(b)}{\overline{F}(x) - \overline{F}(b)} = -\frac{xf(x) - bf(b)}{\overline{F}(x)} = \frac{x\overline{F}'(x)}{\overline{F}(x)} + \frac{bf(b)}{\overline{F}(x)},$$

are also decreasing on $[a, b]$. Since $F$ is increasing and $\overline{F}$ is decreasing, it follows that 

$$x \mapsto xF'(x)/F(x)$$

and 

$$x \mapsto x\overline{F}'(x)/\overline{F}(x)$$

are also decreasing on $[a, b]$, that is, $F$ and $\overline{F}$ are geometrically concave. QED.
Some special class of distributions

- **IFR** (increasing failure rate): an absolutely continuous distribution (with support \((0, \infty)\)) is in the class IFR if the hazard function \(x \mapsto R(x) = -\ln F(x)\) is convex, that is, the failure rate

\[
x \mapsto r(x) = R'(x) = f(x)/F(x) = -F'(x)/F(x)
\]

is increasing on \((0, \infty)\). Equivalently, an absolutely continuous distribution (with support \((0, \infty)\)) is in the class IFR if for all \(y > 0\) the function \(x \mapsto \bar{F}(x + y)/\bar{F}(x)\) is decreasing on \((0, \infty)\). The class **DFR** (decreasing failure rate) is defined analogously: such kind of distributions for which \(x \mapsto \bar{F}(x + y)/\bar{F}(x) = P(X \geq x + y | X \geq x)\) is decreasing for any \(y > 0\).

- **NBU** (new is better than used): a distribution is in this class if for each \(x, y > 0\) we have \(\bar{F}(x + y) \leq \bar{F}(x)\bar{F}(y)\). Since the above inequality can be rewritten as

\[
\bar{F}(x + y)/\bar{F}(x) = P(X \geq x + y | X \geq x) \leq P(X \geq y) = \bar{F}(y),
\]

in the case of a **NBU** distribution the chances (to survive) of a newborn individual are better than of another individual who lived already some time. It is important to note that a distribution is **IFR** if and only if its survival function is log-concave. Moreover, it can be shown that **IFR \(\subseteq\) NBU**.
Suppose that a cancer specialist believes that a patient newly diagnosed as having stage 1 neck cancer has a smaller chance of survival than does a patient who, following a similar initial diagnosis, has received the treatment (say a radiation implant) and then has survived 5 years. How can such a claim be tested?


**IPFR** (increasing proportionate failure rate): a distribution is in this class if the function

\[ x \mapsto xr(x) = xf(x)/\bar{F}(x) = -x\bar{F}'(x)/\bar{F}(x) \]

is increasing on \((0, \infty)\). Observe that this condition means exactly that \(\bar{F}\) is geometrically concave, of which sufficient condition is that \(f\) is geometrically concave, according to the previous theorem. Note that the expression \(x \mapsto 1 + xf'(x)/f(x)\) is called ISE (income-share elasticity) in the literature. It can be shown that **IFR\subset IPFR**.

Convexity with respect to power means

The Hölder means (or power means) are associated to the generating function \( \varphi_p : (0, \infty) \to \mathbb{R} \), defined by

\[
\varphi_p(x) = \begin{cases} 
  x^p, & \text{if } p \neq 0 \\
  \ln x, & \text{if } p = 0,
\end{cases}
\]

and have the following form

\[
M_{\varphi_p}^{(\lambda)}(x, y) = \begin{cases} 
  \left[ \lambda x^p + (1 - \lambda) y^p \right]^{1/p}, & \text{if } p \neq 0 \\
  x^{\lambda} y^{1-\lambda}, & \text{if } p = 0.
\end{cases}
\]

Now, let \( p \) and \( q \) be two arbitrary real numbers. We say that a function \( f : [a, b] \subseteq (0, \infty) \to (0, \infty) \) is \((M_{\varphi_p}, M_{\varphi_q})\)–convex, or simply \((p, q)\)–convex, if the inequality

\[
f(M_{\varphi_p}^{(\lambda)}(x, y)) \leq M_{\varphi_q}^{(\lambda)}(f(x), f(y))
\]

is valid for all \( p, q \in \mathbb{R}, x, y \in [a, b] \) and \( \lambda \in [0, 1] \), where \( \varphi_q \) is defined by

\[
\varphi_q(x) = \begin{cases} 
  x^q, & \text{if } q \neq 0 \\
  \ln x, & \text{if } q = 0.
\end{cases}
\]

If the above inequality is reversed, then we say that the function \( f \) is \((M_{\varphi_p}, M_{\varphi_q})\)–concave, or simply \((p, q)\)–concave. Observe that the \((1, 1)\)–convexity is the usual convexity, the \((1, 0)\)–convexity is exactly the log-convexity, while the \((0, 0)\)–convexity corresponds to the case of the geometrical convexity.
Characterization of the convexity with respect to power means

The following preliminary result gives us a characterization of differentiable \((p, q)\)-convex functions.

**Lemma**

Let \( p, q \in \mathbb{R} \) and let \( f : [a, b] \subseteq (0, \infty) \to (0, \infty) \) be a differentiable function. The function \( f \) is \((p, q)\)-convex ((\(p, q\))-concave) if and only if \( x \mapsto x^{1-p} f'(x)[f(x)]^{q-1} \) is increasing (decreasing).

The next result generalizes the above theorems about the preservation of the log-convexity and geometrically convexity.

**Theorem**

Let \( f : [a, b] \subseteq (0, \infty) \to (0, \infty) \) be a continuously differentiable probability density function. If \( p \in [0, 1] \), then the following implications are true:

(a) \( f \) is \((p, 0)\)-concave \( \Rightarrow \) \( F \) is \((p, 0)\)-concave \( \Rightarrow \) \( F_l \) is \((p, 0)\)-concave.

(b) \( f \) is \((p, 0)\)-concave \( \Rightarrow \) \( \bar{F} \) is \((p, 0)\)-concave \( \Rightarrow \) \( \bar{F}_r \) is \((p, 0)\)-concave.

(c) \( f \) is \((p, 0)\)-concave \( \Rightarrow \) \( x \mapsto x^{1-p} r(x) \) is increasing.

Moreover, if \( p \notin (0, 1) \), then the following implication also hold:

(d) \( a^{1-p} f(a) = 0 \) and \( f \) is \((p, 0)\)-convex \( \Rightarrow \) \( F \) is \((p, 0)\)-convex \( \Rightarrow \) \( F_l \) is \((p, 0)\)-convex.

(e) \( b^{1-p} f(b) = 0 \) and \( f \) is \((p, 0)\)-convex \( \Rightarrow \) \( \bar{F} \) is \((p, 0)\)-convex \( \Rightarrow \) \( \bar{F}_r \) is \((p, 0)\)-convex.

(f) \( b^{1-p} f(b) = 0 \) and \( f \) is \((p, 0)\)-convex \( \Rightarrow \) \( x \mapsto x^{1-p} r(x) \) is decreasing.

Geometrical concavity of some common univariate distributions

Standard normal distribution

The standard normal distribution is a normal distribution with mean 0 and standard deviation 1. This distribution has support \([a, b] = \mathbb{R}\), pdf \(\varphi : \mathbb{R} \to (0, \infty)\), cdf \(\Phi : \mathbb{R} \to (0, 1)\) and survival function \(\overline{\Phi} : \mathbb{R} \to (0, \infty)\), defined by

\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt
\]

and

\[
\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} \, dt.
\]

Clearly, \(\zeta(x) = f'(x)/f(x) = -x, \zeta''(x) = -1\) and \(\zeta'''(x) = 0\) for all \(x \in \mathbb{R}\). Since \(\zeta\) is decreasing, the functions \(f, \Phi\) and \(\overline{\Phi}\) are log-concave. Moreover, since \(x \mapsto x\zeta(x) = -x^2\) is decreasing on \([0, \infty)\), it follows that \(\varphi\) is geometrically concave on \([0, \infty)\). Consequently, the functions \(\Phi\) and \(\overline{\Phi}\) are geometrically concave too on \([0, \infty)\). More generally, since \(\varphi\) is \((p, 0)\)–concave on \([0, \infty)\) for all \(p \leq 2\) and \((p, 0)\)–convex for all \(p \geq 2\), it follows that the functions \(\Phi\) and \(\overline{\Phi}\) are \((p, 0)\)–concave on \([0, \infty)\) for all \(p \in [0, 1]\) and \(\overline{\Phi}\) is \((p, 0)\)–convex on \([0, \infty)\) for all \(p \geq 2\). Moreover, since \(\Phi\) is increasing and \(\overline{\Phi}\) is decreasing, it follows that \(\Phi\) is \((p, 0)\)–concave on \([0, \infty)\) for all \(p \geq 0\) and \(\overline{\Phi}\) is \((p, 0)\)–concave on \([0, \infty)\) for all \(p \leq 0\).
The Student $t$ distribution

This distribution has support $\mathbb{R}$ and pdf, cdf, respectively, defined by

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu \pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

and

$$F(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu \pi} \Gamma\left(\frac{\nu}{2}\right)} \int_{-\infty}^{x} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \, dt = \frac{1}{2} + x \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu \pi} \Gamma\left(\frac{\nu}{2}\right)} \cdot \, _2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}, \frac{3}{2}, -\frac{x^2}{\nu}\right),$$

where $\nu > 0$ is the degree of freedom, $\Gamma$ is the Euler gamma function, and $\, _2F_1$ stands for the Gaussian hypergeometric function. In this case we have

$$\zeta'(x) = -\left[\frac{(\nu + 1)x}{x^2 + \nu}\right]' = -(\nu + 1) \frac{\nu - x^2}{(x^2 + \nu)^2}$$

and

$$[x\zeta(x)]' = -\left[\frac{(\nu + 1)x^2}{x^2 + \nu}\right]' = -(\nu + 1) \frac{2x\nu}{(x^2 + \nu)^2}.$$
More generally, since
\[
\left[ x^{1-p} \zeta(x) \right]' = -\left[ \frac{(\nu + 1)x^{2-p}}{x^2 + \nu} \right]' = -(\nu + 1) \frac{x^{1-p} [\nu(2 - p) - px^2]}{(x^2 + \nu)^2},
\]
it follows that \( f \) is \((p, 0)-\)concave on \([0, \infty)\) for all \( \nu > 0 \) and \( p \leq 0 \), and is \((p, 0)-\)convex on \([0, \infty)\) for all \( \nu > 0 \) and \( p \geq 2 \). However, part (d) of the above main Theorem cannot be applied, because \( f(x)/x^{p-1} \) tends to infinity as \( x \) tends to zero. All the same, since \( F \) is increasing and geometrically concave, we obtain that \( F \) is \((p, 0)-\)concave on \([0, \infty)\) for all \( p \geq 0 \). Similarly, since \( \bar{F} \) is decreasing, it is \((p, 0)-\)concave on \([0, \infty)\) for all \( p \leq 0 \). Moreover, by using part (e) of the above main Theorem, we obtain that \( \bar{F} \) is \((p, 0)-\)convex on \([0, \infty)\) for all \( p \geq 2 \).

It is worth mentioning that, since \( f \) is neither log-concave nor log-convex, the corresponding theorem cannot be applied to deduce the log-concavity of the cdf \( F \). Bagnoli and Bergstrom have pointed out, based on numerical experiments, that the cdf is neither log-convex nor log-concave for \( \nu \in \{1, 2, 3, 4, 24\} \). However, as we have pointed out above, the cdf is log-concave on \([0, \infty)\) for all \( \nu > 0 \). Thus, sometimes it is more convenient to study the geometrical concavity (convexity) of the pdf instead of log-concavity (log-convexity). As we will see below, the same situation appears for the Fisher-Snedecor \( F \) distribution.
The Fisher-Snedecor $F$ distribution

This distribution has support $(0, \infty)$ and pdf $f$, defined by

$$f(x) = \frac{\alpha^{\alpha/2} \beta^{\beta/2} x^{\alpha/2-1}}{B(\alpha/2, \beta/2)(\beta + \alpha x)^{(\alpha+\beta)/2}},$$

where $\alpha, \beta > 0$ and $B$ stands for the Euler beta function. The $F$-distribution arises frequently as the null distribution of a test statistic, especially in likelihood-ratio tests, perhaps most notably in the analysis of variance. It is known that if $\alpha > 2$, then $f$ is neither log-concave nor log-convex on the whole interval $(0, \infty)$. Moreover, if $\alpha \leq 2$, then $f$ is log-convex and so is the corresponding survival function $F$. However, there is no information about the log-concavity (log-convexity) of the corresponding cdf. It is easy to verify that for all $\alpha, \beta > 0$ the pdf $f$ and hence the cdf and survival function are geometrically concave on $(0, \infty)$, since the function

$$x \mapsto x \zeta(x) = \frac{xf'(x)}{f(x)} = \frac{\alpha}{2} - 1 - \frac{\alpha + \beta}{2} \frac{\alpha x}{\beta + \alpha x}$$

is clearly decreasing on $(0, \infty)$ for all $\alpha, \beta > 0$. Now, since the cdf is increasing, it follows that it is $(p, 0)$-concave on $(0, \infty)$ for all $\alpha, \beta > 0$ and $p \geq 0$, and hence log-concave on $(0, \infty)$ for all $\alpha, \beta > 0$. 

Árpád Baricz (Babeş-Bolyai University)
Generalization of Minkowski’s inequality

In 1950 Mulholland [Mu] proved the following generalization of the classical Minkowski inequality:

If \( \varphi : (0, \infty) \to (0, \infty) \) is a continuous function with \( \varphi(0^+) = 0 \), and if \( \varphi \) is convex and geometrically convex, then for all \( x_1, x_2, y_1, y_2 > 0 \) we have

\[
\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leq \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2)).
\]

(2)

In 1984 Tardiff [Ta] proved another result of this type: if \( \varphi : (0, \infty) \to (0, \infty) \) is a differentiable function with \( \varphi(0^+) = 0 \), and if \( \varphi \) is increasing and convex, while \( \varphi' \) is geometrically convex, then for all \( x_1, x_2, y_1, y_2 > 0 \) the inequality (2) holds. In 1999 on the 37th International Symposium on Functional Equations (see [Sk]) Schweizer posed the problem on comparing Mulholland’s and Tardiff’s results, i.e. to study the relationship between the geometrical convexity of a function and its derivative. More precisely, if the function \( \varphi : (0, \infty) \to (0, \infty) \) with \( \varphi(0^+) = 0 \) is differentiable, increasing and convex, then there is any relation between the geometrical convexity of \( \varphi \) and \( \varphi' \)?

In 2002 Jarczyk and Matkowski [JM] proved the following result, which gives an answer to the problem of Schweizer.

**Theorem**

Let \( \varphi : (0, \infty) \to (0, \infty) \) be a differentiable function with \( \varphi(0^+) = 0 \). If \( \varphi \) is increasing and \( \varphi' \) is geometrically convex, then \( \lim_{x \to 0} x\varphi'(x) = 0 \) and \( \varphi \) is geometrically convex too.

---


First observe that the above theorem of Jarczyk and Matkowski is close related to the above theorems. Now, let us present an alternative proof of the second assertion of the above Theorem. Since $\varphi'$ is geometrically convex, it follows that the function

$$x \mapsto 1 + \frac{x \varphi''(x)}{\varphi'(x)} = \frac{(x \varphi'(x))'}{\varphi'(x)}$$

is increasing, and applying the monotone form of l'Hospital's rule, the function

$$x \mapsto \frac{x \varphi'(x)}{\varphi(x)} = \frac{x \varphi'(x) - \lim_{x \to 0} x \varphi'(x)}{\varphi(x) - \lim_{x \to 0} \varphi(x)}$$

is increasing too, and consequently the function $\varphi$ is geometrically convex too.
Montel’s result on geometrical convexity

In 1928 Montel [Mo] proved the following result: Let \( f : [0, a) \rightarrow [0, \infty) \) be a continuous function. If \( f \) is geometrically convex on \((0, a)\), then the function

\[
x \mapsto \int_0^x f(t) \, dt
\]

is also continuous on \([0, a)\) and is geometrically convex on \((0, a)\).

Motivated by Montel’s result, recently Zhang and Chu [ZC] proved the following.

**Theorem**

Let \( f : [a, b] \subseteq (0, \infty) \rightarrow (0, \infty) \) be a geometrically concave function. Then the functions

\[
x \mapsto \int_a^x f(t) \, dt \quad \text{and} \quad x \mapsto \int_x^b f(t) \, dt
\]

are geometrically concave too. Moreover, if \( f \) is a twice differentiable geometrically convex function and for all \( x \in [a, b] \) we have \( xf’(x) + f(x) > 0 \), then the function

\[
x \mapsto \int_a^x f(t) \, dt + \frac{af^2(a)}{f(a) + af’(a)}
\]

is geometrically convex too.


Theorem

Let \( f : [a, b] \subseteq (0, \infty) \to (0, \infty) \) be a differentiable function and let \( g \) and \( h \) be the left-hand and right-hand integral of \( f \), i.e.

\[
g(x) = \int_a^x f(t) \, dt \quad \text{and} \quad h(x) = \int_x^b f(t) \, dt.
\]

Then the following assertions are true:

(a) If for all \( p \in [0, 1] \) the function \( f \) is \((p, 0)\)-concave, then the function \( g \) is \((p, q)\)-concave for all \( p \in [0, 1] \) and \( q \leq 0 \). If, in addition the function \( x \mapsto x^{1-p}f(x) \) is increasing for all \( p \in [0, 1] \), then \( g \) is \((p, q)\)-concave for all \( p \in [0, 1] \) and \( q \in (0, 1) \). Moreover, if for all \( p \in \mathbb{R} \) the function \( x \mapsto x^{1-p}f(x) \) is increasing, then \( g \) is \((p, q)\)-convex for all \( p \in \mathbb{R} \) and \( q \geq 1 \).

(b) If for all \( p \in [0, 1] \) the function \( f \) is \((p, 0)\)-concave, then the function \( h \) is \((p, q)\)-concave for all \( p \in [0, 1] \) and \( q \leq 0 \). If, in addition the function \( x \mapsto x^{1-p}f(x) \) is decreasing for all \( p \in [0, 1] \), then \( g \) is \((p, q)\)-concave for all \( p \in [0, 1] \) and \( q \in (0, 1) \). Moreover, if for all \( p \in \mathbb{R} \) the function \( x \mapsto x^{1-p}f(x) \) is decreasing, then \( h \) is \((p, q)\)-convex for all \( p \in \mathbb{R} \) and \( q \geq 1 \).

(c) If for all \( p \notin (0, 1) \) we have \( a^{1-p}f(a) = 0 \) and the function \( f \) is \((p, 0)\)-convex, then \( g \) is \((p, q)\)-convex for all \( p \notin (0, 1) \) and \( q \geq 0 \). If, in addition the function \( x \mapsto x^{1-p}f(x) \) is increasing for all \( p \notin (0, 1) \), then \( g \) is \((p, q)\)-convex for all \( p \notin (0, 1) \) and \( q < 0 \).

(d) If for all \( p \notin (0, 1) \) we have \( b^{1-p}f(b) = 0 \) and the function \( f \) is \((p, 0)\)-convex, then \( h \) is \((p, q)\)-convex for all \( p \notin (0, 1) \) and \( q \geq 0 \). If, in addition the function \( x \mapsto x^{1-p}f(x) \) is decreasing for all \( p \notin (0, 1) \), then \( h \) is \((p, q)\)-convex for all \( p \notin (0, 1) \) and \( q < 0 \).