Turán type inequalities for modified Bessel functions of the first and second kind

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Abstract

Motivated by some applications in applied mathematics, biology, chemistry, physics and engineering sciences, new tight Turán type inequalities for modified Bessel functions of the first and second kind are deduced. These inequalities provide sharp lower and upper bounds for the Turánian of modified Bessel functions of the first and second kind, and in most cases the relative errors of the bounds tend to zero as the argument tends to infinity. The chief tools in our proofs are some ideas of Gronwall, an integral representation of Ismail for the quotient of modified Bessel functions of the second kind, results of Hartman and Watson and some recent results of Segura. As an application of the main results it is shown that the product of modified Bessel functions of the first and second kind is strictly geometrically concave.
In the 1940’s, while studying the zeros of Legendre polynomials, the Hungarian mathematician Paul Turán [Tu] discovered the inequality

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0,$$

which holds for all $|x| \leq 1$ and $n \in \{1, 2, 3, \ldots\}$. Here, $P_n$ is the Legendre polynomial, that is,

$$P_n(x) = \frac{d^n}{dx^n} \left( \frac{(x^2 - 1)^n}{n!2^n} \right) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n - 2k)!}{k!(n - k)!(n - 2k)!} x^{n-2k}. $$
Turán’s inequality for Legendre polynomials

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The first six Legendre polynomials are as follows:

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \]
\[ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \]
The first six Legendre polynomials

Turán's inequality for Legendre polynomials
Turán type inequalities for modified Bessel functions

References

Árpád Baricz
Turán inequalities for modified Bessel functions
The Turán expression \( P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \) for \( n \in \{1, 2, 3, 4\} \).
Nowadays Turán’s inequality has an extensive bibliography (there are at least 200 papers on this subject), and has been extended in many directions for various orthogonal polynomials and special functions, for example: Laguerre, Hermite, ultraspherical, Pollaczek, Lommel, Askey-Wilson, Bernoulli and Appell polynomials, Bessel and modified Bessel functions, Gauss and confluent hypergeometric functions, generalized hypergeometric functions, probability distribution functions, polygamma function and Riemann zeta function, zeros of general Bessel functions, zeros of ultraspherical, Laguerre and Hermite polynomials, gamma function, generalized Marcum $Q$–function, and this list is far from being complete. Some of the results have been applied successfully in problems that arise in information theory, economic theory and biophysics, and motivated by these applications, the Turán type inequalities have recently come under the spotlight once again.
The following anecdote from Askey’s paper [As] about Turán reveals that he used the inequality as his visiting card:

– B. Meulenbeld told me a story about Turán which illustrates Turán’s playfulness. They were in the same compartment of a train, but did not know each other. After some pleasant words, they discovered they were both mathematicians. After Meulenbeld introduced himself and said a little about what he did, it was Turán’s turn to do the same. He took out a piece of paper and wrote

\[ P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geq 0, \]

and asked Meulenbeld if he knew this. Meulenbeld said, “Of course, that is Turán’s inequality.” Turán then said “I am Turán.” – Árpád Baricz
Definition of modified Bessel functions

Let us denote by $I_\nu$ and $K_\nu$ the modified Bessel functions of the first and second kinds of real order $\nu$ (see [Wa]), which are the linearly independent particular solutions of the homogeneous second order modified Bessel differential equation [Wa, p. 77]

$$x^2 y''(x) + xy'(x) - \left(x^2 + \nu^2\right)y(x) = 0.$$ 

Recall that the modified Bessel function $I_\nu$ has the infinite series representation [Wa, p. 77]

$$I_\nu(x) = \sum_{n \geq 0} \frac{(x/2)^{2n+\nu}}{n! \Gamma(n + \nu + 1)},$$

where $\nu \neq -1, -2, \ldots$ and $x \in \mathbb{R}$, while the modified Bessel function of the second kind $K_\nu$ (called also as the MacDonald or Hankel function), is defined as [Wa, p. 78]

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi},$$

where the right-hand side of this equation is replaced by its limiting value if $\nu$ is an integer or zero.
Here $\Gamma : \mathbb{R} \setminus \{-Z\} \to \mathbb{R}$, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

stands for the well-known Euler gamma function, which satisfies $\Gamma(x + 1) = x\Gamma(x)$ for $x > 0$ and $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}$. 

We note that in view of the above series representation $I_\nu(x) > 0$ for all $\nu > -1$ and $x > 0$. Similarly, by using the integral representation $K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t) dt$, which holds for each $x > 0$ and $\nu \in \mathbb{R}$, one can see that $K_\nu(x) > 0$ for all $x > 0$ and $\nu \in \mathbb{R}$. 

Árpád Baricz

Turán inequalities for modified Bessel functions
Here $\Gamma : \mathbb{R} \setminus \{-\mathbb{Z}\} \to \mathbb{R}$, defined by

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We note that in view of the above series representation $I_\nu(x) > 0$ for all $\nu > -1$ and $x > 0$. Similarly, by using the integral representation [Wa, p. 181]

$$K_\nu(x) = \int_0^{\infty} e^{-x \cosh t} \cosh(\nu t) dt,$$

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Árpád Baricz
The graph of the functions $I_m$ for $m \in \{0, 1, 2, 3\}$. 
The graph of the functions $K_m$ for $m \in \{0, 1, 2, 3\}$. 
Theorem

Let \( I_\nu \) be the modified Bessel functions of the first kind. Then the following Turán-type inequalities hold for all \( \nu > -1 \) and \( x \in \mathbb{R} \)

\[
0 \leq I_\nu^2(x) - I_{\nu-1}(x)I_{\nu+1}(x) \leq I_\nu^2(x)/(\nu + 1). \tag{1}
\]

In each of the above inequalities equality holds if and only if \( x = 0 \). Moreover, these inequalities are sharp in the sense that the constants \( \alpha_\nu = 0 \) and \( \beta_\nu = 1/(\nu + 1) \) are the best possible such that the inequalities

\[
\alpha_\nu I_\nu^2(x) \leq I_\nu^2(x) - I_{\nu-1}(x)I_{\nu+1}(x) \leq \beta_\nu I_\nu^2(x)
\]

hold for all \( x \in \mathbb{R} \) and \( \nu > -1 \).
The graph of the functions $x \mapsto 1 - l_{m-1}(x)l_{m+1}(x)/l_m^2(x)$ for $m \in \{1, 2, 3, 4\}$. 
(1) was deduced first in 1951 by Thiruvenkatachar and Nanjundiah [TN]. The left-hand side of (1) was proved also by Amos [Am] in 1974, and later by Joshi and Bissu [JB] in 1991. Note that the function $\nu \mapsto I_{\nu+a}(x)/I_{\nu}(x)$ is decreasing for each fixed $a \in (0, 2]$ and $x > 0$, where $\nu > -1$ and $\nu \geq -(a + 1)/2$. Consequently, the function $\nu \mapsto I_{\nu}(u)$ is log-concave on $(-1, \infty)$. See [Ba3]. However, this implies just that the left-hand side of (1) holds for all $\nu > 0$. All the same, in 1994 Lorch [Lo] proved that the left-hand side of (1) holds true for all $\nu > -1/2$. Moreover, Lorch [Lo] conjectured that $I_{\nu}^2(x) - I_{\nu-a}(x)I_{\nu+a}(x) > 0$ for all $x > 0$, $a \in (0, 1]$ and $\nu \in (-1, -1/2]$. This conjecture has been verified recently by Kalmykov and Karp [KK], and another proof of (1) was obtained recently by Segura [Se].
For $\nu > -1$ let us consider the function $I_\nu : \mathbb{R} \to [1, \infty)$, defined by

$$I_\nu(x) = 2^\nu \Gamma(\nu + 1)x^{-\nu} l_\nu(x) = \sum_{n \geq 0} \frac{(1/4)^n}{(\nu + 1)_n n!} x^{2n},$$

where $(\nu + 1)_n = (\nu + 1)(\nu + 2)\ldots(\nu + n) = \Gamma(\nu + n + 1)/\Gamma(\nu + 1)$ is the well-known Pochhammer (or Appell) symbol defined in terms of Euler’s gamma function, and $l_\nu$ is the modified Bessel function of the first kind.
For \( \nu > -1 \) let us consider the function \( I_\nu : \mathbb{R} \to [1, \infty) \), defined by

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I_\nu(x) = 2^\nu \Gamma(\nu + 1)x^{-\nu}I_\nu(x) = \sum_{n \geq 0} \frac{(1/4)^n}{(\nu + 1)_n n!} x^{2n},
\]

where \((\nu + 1)_n = (\nu + 1)(\nu + 2)\ldots(\nu + n) = \Gamma(\nu + n + 1)/\Gamma(\nu + 1)\) is the well-known Pochhammer (or Appell) symbol defined in terms of Euler’s gamma function, and \( I_\nu \) is the modified Bessel function of the first kind.

The right-hand side of (1) can be generalized as follows (see [Ba1]):

**Theorem**

*The function \( \nu \mapsto I_\nu(x) \) is log-convex on \((-1, \infty)\) for all \( x \in \mathbb{R} \) fixed. In particular, the Turán-type inequality

\[
I^2_\nu(x) \leq I_{\nu - 1}(x)I_{\nu + 1}(x)
\]

or equivalently

\[
I^2_\nu(x) - I_{\nu - 1}(x)I_{\nu + 1}(x) \leq I^2_\nu(x)/(\nu + 1)
\]

holds for all \( x \in \mathbb{R} \) and \( \nu > 0 \).*
Proof for the log-convexity of $\nu \mapsto \mathcal{I}_\nu(x)$

Consider the infinite product representation of the modified Bessel function of the first kind

$$\mathcal{I}_\nu(x) = \prod_{n \geq 1} \left(1 + \frac{x^2}{j_{\nu,n}^2}\right),$$

where $j_{\nu,n}$ is the $n$th positive zero of the Bessel function $J_\nu$. Thus, we have

$$\log[\mathcal{I}_\nu(x)] = \sum_{n \geq 1} \log \left(1 + \frac{x^2}{j_{\nu,n}^2}\right).$$

Owing to Elbert [EI] it is known that $\nu \mapsto j_{\nu,n}$ is concave on $(-n, \infty)$ for all $n \geq 1$. Consequently, we have that $\nu \mapsto j_{\nu,n}$ and $\nu \mapsto \log j_{\nu,n}$ are concave on $(-1, \infty)$ for all $n \geq 1$. Hence, $\nu \mapsto -2 \log j_{\nu,n}$ is convex, i.e. $\nu \mapsto 1/j_{\nu,n}^2$ is log-convex on $(-1, \infty)$. But this implies that for all $n \geq 1$ the function $\nu \mapsto \log(1 + x^2/j_{\nu,n}^2)$ is convex on $(-1, \infty)$, and consequently the function $\nu \mapsto \log \mathcal{I}_\nu(x)$ is convex too on $(-1, \infty)$ as a sum of convex functions. QED.
Theorem

Let $K_{\nu}$ be the modified Bessel function of the second kind. Then the following Turán type inequalities hold for all $\nu > 1$ and $x > 0$

$$K_{\nu}^2(x)/(1 - \nu) < K_{\nu}^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) < 0.$$ (2)

Moreover, the right-hand side of (2) holds true for all $\nu \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $\alpha_\nu = 1/(1 - \nu)$ and $\beta_\nu = 0$ are the best possible such that the inequalities

$$\alpha_\nu K_{\nu}^2(x) < K_{\nu}^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) < \beta_\nu K_{\nu}^2(x)$$

hold for all $x > 0$, $\nu > 1$ and $\nu \in \mathbb{R}$, respectively.
The graph of the functions $x \mapsto 1 - K_{m-1}(x)K_{m+1}(x)/K_m^2(x)$ for $m \in \{1, 2, 3, 4\}$.
The right-hand side of (2) was proved in 1978 by Ismail and Muldoon [IM]. For $\nu > 1/2$ this inequality was also deduced in 2006 by Laforgia and Natalini [LN1]. Ismail and Muldoon [IM], by using the Nicholson formula concerning the product of two modified Bessel functions of different order, proved that the function $\nu \mapsto K_{\nu+a}(x)/K_\nu(x)$ is increasing on $\mathbb{R}$ for each fixed $x > 0$ and $a > 0$. As Muldoon [Mu] pointed out, this implies that $\nu \mapsto K_\nu(x)$ is log-convex on $\mathbb{R}$ for each fixed $x > 0$. Recently, by using the classical Hölder-Rogers inequality, the author [Ba3] pointed out that the function $\nu \mapsto K_\nu(x)$ is in fact strictly log-convex on $\mathbb{R}$ for each fixed $x > 0$. 
The right-hand side of (2) was proved in 1978 by Ismail and Muldoon [IM]. For \( \nu > 1/2 \) this inequality was also deduced in 2006 by Laforgia and Natalini [LN1]. Ismail and Muldoon [IM], by using the Nicholson formula concerning the product of two modified Bessel functions of different order, proved that the function \( \nu \mapsto K_{\nu+a}(x)/K_\nu(x) \) is increasing on \( \mathbb{R} \) for each fixed \( x > 0 \) and \( a > 0 \). As Muldoon [Mu] pointed out, this implies that \( \nu \mapsto K_\nu(x) \) is log-convex on \( \mathbb{R} \) for each fixed \( x > 0 \). Recently, by using the classical Hölder-Rogers inequality, the author [Ba3] pointed out that the function \( \nu \mapsto K_\nu(x) \) is in fact strictly log-convex on \( \mathbb{R} \) for each fixed \( x > 0 \).

Two different proofs of (2) were deduced recently by Segura [Se] and the author [Ba2].
It is worth to mention that according to the corresponding recurrence relations

\[ I_{\nu-1}(x) = \left( \frac{\nu}{x} \right) I_{\nu}(x) + I'_{\nu}(x) \quad \text{and} \quad I_{\nu+1}(x) = I'_{\nu}(x) - \left( \frac{\nu}{x} \right) I_{\nu}(x), \]

\[ K_{\nu-1}(x) = -\left( \frac{\nu}{x} \right) K_{\nu}(x) - K'_{\nu}(x) \quad \text{and} \quad K_{\nu+1}(x) = -K'_{\nu}(x) + \left( \frac{\nu}{x} \right) K_{\nu}(x), \]

the left-hand side of the Turán type inequality (1) is equivalent to

\[ \frac{x I'_{\nu}(x)}{I_{\nu}(x)} < \sqrt{x^2 + \nu^2}, \] (3)

while the right-hand side of (2) is equivalent to

\[ \frac{x K'_{\nu}(x)}{K_{\nu}(x)} < -\sqrt{x^2 + \nu^2}. \] (4)

Moreover, the inequalities (3) and (4) together imply that

\[ x \left[ \log(P_\nu(x)) \right]' = x \left[ \log(I_{\nu}(x)K_{\nu}(x)) \right]' < 0, \]

which implies that the function \( x \mapsto P_\nu(x) = I_{\nu}(x)K_{\nu}(x) \) is strictly decreasing on \((0, \infty)\) for all \( \nu > -1 \).
The graph of the functions $P_m$ for $m \in \{0, 1, 2, 3\}$.
Note that the above monotonicity property of $P_\nu$ was proved earlier by Penfold et al. \cite{PVG} and was motivated by a problem in biophysics. For the sake of completeness we recall also that the inequality (3) was deduced first\footnote{To prove (3) Gronwall \cite{Gron} claimed that the function $x \mapsto \sqrt{x^2 + \nu^2} - x l'_\nu(x)/l_\nu(x)$ is increasing on $(0, \infty)$ for all $\nu > 0$. However, this claim is not true. All the same, the inequality (3) is valid, and it follows from the fact that the function $x \mapsto x l'_\nu(x)/l_\nu(x) - \nu$ is increasing on $(0, \infty)$ for all $\nu > 0$.} by Gronwall \cite{Gron} for $\nu > 0$, motivated by a problem in wave mechanics. This inequality was deduced also for $\nu \in \{1, 2, \ldots\}$ by Phillips and Malin \cite{PM}, and for $\nu > 0$ by Amos \cite{Am} and Paltsev \cite{Pa}. The inequality (4) was deduced first for $\nu \in \{1, 2, \ldots\}$ by Phillips and Malin \cite{PM}, and later for $\nu \geq 0$ by Paltsev \cite{Pa}. We note that the Turán type inequalities (1), (2), (3) and (4) as well as the monotonicity of the product of $P_\nu$ were used in various problems related to modified Bessel functions in various topics of applied mathematics, biology, chemistry and physics.
The monotonicity of $P_\nu$ for $\nu > 1$ is used (without proof) in some papers of Radwan et al. [RDH], [RH] about the hydrodynamic and hydromagnetic instability of different cylindrical models, and also in the paper of Hasan [Has], where the electrogravitational instability of nonoscillating streaming fluid cylinder under the action of the selfgravitating, capillary and electrodynamic forces has been discussed. In these papers the authors use (without proof) the inequality

$$P_\nu(x) < \frac{1}{2}$$

for all $\nu > 1$ and $x > 0$. We note that the above inequality readily follows from the fact that $P_\nu$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$. More precisely, for all $x > 0$ and $\nu > 1$ we have

$$P_\nu(x) < \lim_{x \to 0^+} P_\nu(x) = \frac{1}{2\nu} < \frac{1}{2}.$$
The monotonicity of $P_\nu$ for $\nu > 1$ is used (without proof) in some papers of Radwan et al. [RDH], [RH] about the hydrodynamic and hydromagnetic instability of different cylindrical models, and also in the paper of Hasan [Has], where the electrogravitational instability of nonoscillating streaming fluid cylinder under the action of the selfgravitating, capillary and electrodynaminc forces has been discussed. In these papers the authors use (without proof) the inequality

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$$P_\nu(x) < \lim_{x \to 0} P_\nu(x) = \frac{1}{2\nu} < \frac{1}{2}.$$

The Turán type inequality (1) and the right-hand side of (2), together with the monotonicity of $P_\nu$ were used, among other things, by Klimek and McBride [KM] to prove that a Dirac operator, subject to Atiyah-Patodi-Singer-like boundary conditions on the solid torus, has a bounded inverse, which is actually a compact operator.
Recently, Simitev and Biktashev [SB] used the fact that the function $x \mapsto xI'_\nu(x)/I_\nu(x)$ is increasing on $(0, \infty)$ together with the inequality (3) in the study of asymptotic restitution curves in the caricature Noble model of electrical excitation in the heart. As it was pointed out above the inequality (3) is equivalent to the left-hand side of the Turán type inequality (1). Moreover, because of

$$x \left[ I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) \right] = I^2_\nu(x) \left[ \frac{xI'_\nu(x)}{I_\nu(x)} \right]' \quad (5)$$

the fact that the function $x \mapsto xI'_\nu(x)/I_\nu(x)$ is increasing is also equivalent to the left-hand side of the Turán type inequality (1). Thus, Simitev and Biktashev [SB] actually used two times in their study exactly the left-hand side of the Turán type inequality (1).
Very recently, in order to prove that \( [xI_{\nu+1}(x)/I_{\nu}(x)]' > 0 \) for all \( \nu \geq 0 \) and \( x > 0 \), Schlenk and Sicbaldi [SS] rediscovered the left-hand side of (1). They used

\[
\left( \frac{xI_{\nu+1}(x)}{I_{\nu}(x)} \right)' = \frac{x \left[ I_{\nu}^2(x) - I_{\nu-1}(x)I_{\nu+1}(x) \right]}{I_{\nu}(x)},
\]

which in view of the recurrence relation \( xI_{\nu}'(x) = \nu I_{\nu}(x) + xI_{\nu+1}(x) \), is actually the same as (5). We also mention that Schlenk and Sicbaldi [SS] rediscovered also the corresponding Turán type inequality for Bessel functions of the first kind. These results on Bessel and modified Bessel functions of the first kind were used to study bifurcating extremal domains for the first eigenvalue of the Laplacian. More precisely, the Turán type inequalities were used in the study of the monotonicity of the first eigenvalue of a linearized operator, in order to show that this operator satisfies the assumptions of the Crandall-Rabinowitz theorem, implying the main result of Schlenk and Sicbaldi’s paper [SS].
Let $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open disk with center $z_0 \in \mathbb{C}$ and radius $r > 0$ and let us denote the particular disk $D(0, 1)$ by $D$. Moreover, let $A$ be the class of analytic functions defined in the unit disk $D$, which can be normalized as $f(z) = z + a_2z^2 + \ldots$, that is, $f(0) = f'(0) - 1 = 0$. The class of starlike functions, denoted by $S^*$, is the subclass of $A$ which consists of functions $f$ for which the domain $f(D)$ is starlike with respect to 0. An analytic description of $S^*$ is

$$S^* = \left\{ f \in A \left| \Re \left[ \frac{zf'(z)}{f(z)} \right] > 0 \text{ for all } z \in D \right. \right\}.$$

Moreover, consider the class of starlike functions of order $\beta \in [0, 1)$, that is,

$$S^*(\beta) = \left\{ f \in A \left| \Re \left[ \frac{zf'(z)}{f(z)} \right] > \beta \text{ for all } z \in D \right. \right\}.$$
The real numbers

\[ r^*(f) = \sup \left\{ r > 0 \mid \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0 \text{ for all } z \in D(0, r) \right\} \]

and

\[ r^*_\beta(f) = \sup \left\{ r > 0 \mid \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > \beta \text{ for all } z \in D(0, r) \right\}, \]

are called the radius of starlikeness and the radius of starlikeness of order \( \beta \) of the function \( f \), respectively. We note that in fact \( r^*(f) \) is the largest radius such that \( f(D(0, r^*(f))) \) is a starlike domain with respect to 0.
The real numbers

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and

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By using the fact that \( x \mapsto x l'_\nu(x) / l_\nu(x) \) is increasing on \((0, \infty)\) for all \( \nu > -1 \), it can be proved that if \( \nu \in (-1, 0) \), then \( r^*_\beta(f_\nu) = x_{\nu, \beta} \), where \( f_\nu(z) = [2^\nu \Gamma(\nu + 1) J_\nu(z)]^{1/\nu} \) and \( x_{\nu, \beta} \) denotes the unique positive root of the equation \( x l'_\nu(x) - \beta \nu l_\nu(x) = 0 \). See [BKS] for more details.
Note that, as it was pointed out by Segura \[Se\], the left-hand side of the Turán type inequality (2) provides actually an upper bound for the effective variance of the generalized Gaussian distribution. More precisely, Alexandrov and Lacis \[AL\] used (without proof) the inequality \(0 < v_{\text{eff}} < 1/(\mu - 1)\) for \(\mu = \nu + 4\), where

\[
\begin{align*}
v_{\text{eff}} := & \frac{\int_0^\infty r^2 f_{\nu}(r) dr \left[ \int_0^\infty r^4 f_{\nu}(r) dr \right]}{\left[ \int_0^\infty r^3 f_{\nu}(r) dr \right]^2} - 1 = \frac{K_{\mu-1}(1/w)K_{\mu+1}(1/w)}{[K_\mu(1/w)]^2} - 1
\end{align*}
\]

is the effective variance of the generalized Gaussian distribution and

\[
f_{\nu}(r) = \frac{1}{2K_{\nu+1}(1/w)} \frac{r^\nu}{s^{\nu+1}} \exp\left[-\frac{1}{2w} \left( \frac{s}{r} + \frac{r}{s} \right) \right]
\]

is the generalized inverse Gaussian particle size distribution function, \(w\) represents the width of the distribution, \(s\) is an effective size parameter, and \(\nu\) is the order of the distribution.
Recently, motivated by some results in finite elasticity, Laforgia and Natalini [LN2] proved that for \( x > 0 \) and \( \nu \geq 0 \) the following inequality is valid

\[
\frac{I_{\nu}(x)}{I_{\nu-1}(x)} > \frac{-\nu + \sqrt{x^2 + \nu^2}}{x}.
\]  
(6)

We note an alternative proof of (6) was given recently by Kokologiannaki [Ko]. Moreover, it can be shown that the inequality (6) is equivalent to (3), which is equivalent to the left-hand side of (1). Observe that the inequality (6) can be rewritten in the form

\[
\frac{1}{x} \frac{I_{\nu}(x)}{I_{\nu-1}(x)} > \frac{1}{\nu + \sqrt{x^2 + \nu^2}},
\]  
(7)

where \( x > 0 \) and \( \nu \geq 0 \). Segura [Se] has pointed out that the inequality (7), which is actually equivalent to the left-hand side of (1), appears in a problem of chemistry.
More precisely, Lushnikov et al. [LBF] considered the mean number of molecules of a given class dissolved in a water droplet and compared the so-called classical and stochastic approaches. If $n_c$ and $n_s$ are the respective mean numbers of molecules by using the classical and stochastic approaches, then according to Segura [Se], after the redefinition of the variables it can be shown that

$$n_c = \frac{x^2}{4} \frac{1}{\nu + 1 + \sqrt{x^2 + (\nu + 1)^2}} \quad \text{and} \quad n_s = \frac{x}{4} \frac{l_{\nu+1}(x)}{l_{\nu}(x)}$$

and by using the inequality (7) for all $x > 0$ and $\nu \geq -1$ we have $n_s > n_c$. Note that this inequality was known before only for small or large values of $x$. 
It is also interesting to note that the Turánian \( K^2_\nu(x) - K_{\nu-1}(x)K_{\nu+1}(x) \) appears in the variance of the non-central \( F \)-Bessel distribution defined by Thabane and Drekic [TD], related with the variance of a different distribution. We note that the left-hand side of the inequality (1) was used also by Milenkovic and Compton [MC], and for \( \nu = 1 \) by Bertini et al. [BGP]. The property that \( x \mapsto xI'_\nu(x)/I_\nu(x) \) is increasing on \((0, \infty)\) for all \( \nu \geq 0 \) was used by Giorgi and Smits [GS1], [GS2], and also by Lombardo et al. [LMVD] together with its analogue that \( x \mapsto xK'_\nu(x)/K_\nu(x) \) is decreasing on \((0, \infty)\) for all \( \nu \geq 0 \). The later property is used by Lombardo et al. [LMVD] without proof, however, this is actually equivalent to the right-hand side of the Turán type inequality (2), according to relation

\[
x \left[ K^2_\nu(x) - K_{\nu-1}(x)K_{\nu+1}(x) \right] = K^2_\nu(x) \left[ \frac{xK'_\nu(x)}{K_\nu(x)} \right]'.
\] (8)
If \( \nu \geq 1/2 \) and \( x > 0 \), then the next Turán type inequalities are valid

\[
\frac{\nu + \frac{1}{2}}{\nu + 1} \cdot \frac{1}{\sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2}} \cdot I_\nu^2(x) < I_\nu^2(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{\sqrt{x^2 + \nu^2 - \frac{1}{4}}} \cdot I_\nu^2(x).
\]

Moreover, the left-hand side of (9) holds true for all \( \nu \geq -1/2 \) and \( x > 0 \). Each of the above inequalities are sharp as \( x \to \infty \), and the left-hand side of (9) is sharp as \( x \to 0 \).
The graph of the functions $x \mapsto L(x) = \frac{3}{4} \cdot \frac{1}{\sqrt{x^2 + \frac{9}{4}}}$, $x \mapsto 1 - I_0(x)I_2(x)/I_1^2(x)$ and $x \mapsto U(x) = \frac{1}{\sqrt{x^2 + \frac{3}{4}}}$. 
Now, let us consider the function $x \mapsto \lambda_\nu(x) = y_\nu(x) - \sqrt{x^2 + (\nu + 1)^2}$.

Based on numerical experiments we believe, but are unable to prove the following result: if $\nu \geq -1/2$ and $x > 0$, then $\lambda_\nu'(x) > 0$, and equivalently the Turán type inequality

$$
\frac{1}{\sqrt{x^2 + (\nu + 1)^2}} \cdot I_\nu^2(x) < I_\nu^2(x) - I_{\nu-1}(x)I_{\nu+1}(x)
$$

is valid.

Observe that, if the inequality (10) would be valid, then it would improve the left-hand side of (9) for all $\nu \geq -1/2$ and $x > 0$. 
Proof of the right-hand side of (9)

We use the idea of Gronwall [Gron]. Let \( \mu = \nu^2 - 1/4 \) and let \( y_{\nu}(x) = x l'_{\nu}(x)/l_{\nu}(x) \). Observe that, since \( l_{\nu} \) satisfies the modified Bessel differential equation, the function \( y_{\nu} \) satisfies

\[
xy'_{\nu}(x) = x^2 + \nu^2 - y^2_{\nu}(x),
\]

and differentiating both sides of (11) we obtain

\[
xy''_{\nu}(x) = 2x - (2y_{\nu}(x) + 1)y'_{\nu}(x).
\]

Recall also the following inequality of Hartman and Watson [HW]

\[
\frac{xl'_{\nu}(x)}{l_{\nu}(x)} > \sqrt{x^2 + \nu^2 - \frac{1}{4} - \frac{1}{2}},
\]

which is valid for all \( \nu \geq 1/2 \) and \( x > 0 \).
Proof of the right-hand side of (9)

We prove that the function \( x \mapsto u_\nu(x) = \sqrt{x^2 + \mu} - y_\nu(x) \) satisfies \( u'_\nu(x) > 0 \) for all \( \nu \geq 1/2 \) and \( x > 0 \). For this observe that

\[
\sqrt{x^2 + \mu} = \sqrt{\mu} + \frac{x^2}{2\sqrt{\mu}} - \frac{x^4}{8\mu \sqrt{\mu}} + \ldots
\]

and

\[
\sqrt{x^2 + \mu} - y_\nu(x) = \sqrt{\mu} - \nu + \left[ \frac{1}{\sqrt{\mu}} - \frac{1}{\nu + 1} \right] \frac{x^2}{2} - \left[ \frac{1}{\mu \sqrt{\mu}} - \frac{1}{(\nu + 1)^2(\nu + 2)} \right] \frac{x^4}{8} + \ldots,
\]

which implies that for small values of \( x \) the function \( u_\nu \) is strictly increasing.
Proof of the right-hand side of (9)

Thus the first extreme of this function, if any, is a maximum. However, when \( u'_\nu(x) = 0 \), that is,

\[
y'_\nu(x) = \frac{x}{\sqrt{x^2 + \mu}},
\]

by using (12) and (13) we have for all \( x > 0 \) and \( \nu \geq 1/2 \)

\[
u''_\nu(x) = \frac{\mu}{(x^2 + \mu)^{3/2}} + \frac{1 - 2u_\nu(x)}{\sqrt{x^2 + \mu}} > 0,
\]

which is a contradiction. Consequently, the derivative of the function \( u_\nu \) does not vanish, and then \( u'_\nu(x) > 0 \) for all \( \nu \geq 1/2 \) and \( x > 0 \), as we required. This in turn implies that for all \( x > 0 \) and \( \nu \geq 1/2 \) we have

\[
\frac{1}{x} y'_\nu(x) < \frac{1}{\sqrt{x^2 + \mu}},
\]

which in view of (5) is equivalent to the right-hand side of (9). QED.
A $p$-dimensional unit random vector $\mathbf{x}$ ($||\mathbf{x}|| = 1$) is said to have a $p$-variate von Mises-Fisher distribution if its probability density function is given by

$$f(\mathbf{x}) = \frac{\kappa^{\frac{p-1}{2}}}{(2\pi)^{\frac{p}{2}} I_{\frac{p}{2}-1}(\kappa)} \cdot e^{\kappa \mu^T \mathbf{x}}, \quad \mathbf{x} \in S^{p-1},$$

where $||\mu|| = 1$, $\kappa \geq 0$ and $S^{p-1} = \{ \mathbf{x} \in \mathbb{R}^p : ||\mathbf{x}|| = 1 \}$ is the $p$-dimensional unit hypersphere.
A $p$-dimensional unit random vector $\mathbf{x}$ ($||\mathbf{x}|| = 1$) is said to have a $p$-variate von Mises-Fisher distribution if its probability density function is given by

$$f(\mathbf{x}) = \frac{\kappa^{\frac{p}{2}-1}}{(2\pi)^{\frac{p}{2}} I_{\frac{p}{2}-1}(\kappa)} \cdot e^{\kappa \mu^T \mathbf{x}}, \quad \mathbf{x} \in S^{p-1},$$

where $||\mu|| = 1$, $\kappa \geq 0$ and $S^{p-1} = \{ \mathbf{x} \in \mathbb{R}^p : ||\mathbf{x}|| = 1 \}$ is the $p$-dimensional unit hypersphere.

Recently, Tanabe et al. [TFOTI] proposed an iterative algorithm using fixed points to obtain the maximum likelihood estimate for the parameter $\kappa$ of the $p$-variate von Mises–Fisher distribution on the $p$-dimensional unit hypersphere. In their study Tanabe et al. [TFOTI] arrived at the equation

$$\frac{1}{r_{\frac{p}{2}-1}(\hat{\kappa})} = \bar{R},$$

(14)

where $r_{\frac{p}{2}-1}(\hat{\kappa}) = I_{\frac{p}{2}-1}(\hat{\kappa})/I_{\frac{p}{2}}(\hat{\kappa})$, $I_{\nu}$ is the modified Bessel function of the first kind of order $\nu$, and $\bar{R} = ||x_1 + x_2 + \cdots + x_n||/n$ is the mean length of the data vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. 
In order to solve (14) Tanabe et al. [TFOTI] used a fixed point iteration method and for this first derived the following main result: if $\nu \geq 1$, then the function $\Phi_{2\nu} : (0, \infty) \to \mathbb{R}$, defined by

$$\Phi_{2\nu}(x) = R x r_{\nu - 1}(x) = R x \frac{l_{\nu - 1}(x)}{l_\nu(x)},$$

has a unique fixed point.
In order to solve (14) Tanabe et al. [TFOTI] used a fixed point iteration method and for this first derived the following main result: if \( \nu \geq 1 \), then the function \( \Phi_{2\nu} : (0, \infty) \to \mathbb{R} \), defined by

\[
\Phi_{2\nu}(x) = R\chi_{\nu-1}(x) = R\chi \frac{I_{\nu-1}(x)}{I_{\nu}(x)},
\]

has a unique fixed point.

The proof of the main Theorem of Tanabe et al. [TFOTI] was based on the Banach fixed point theorem, and therefore they wanted to prove that the function \( \Phi_{2\nu} \) is a contraction mapping. For this it was enough to prove that \( 0 < \Phi'_{2\nu}(x) < 1 \) for each \( x > 0 \) and \( \nu \geq 1 \). However, the proof of the inequality \( \Phi'_{2\nu}(x) < 1 \) presented in Tanabe et al. [TFOTI] is not correct. All the same, this result is true. See [Ba7] for more details.
For this observe that

$$\Phi'_{2\nu}(x) = \overline{R} \left[ r_{\nu-1}(x) + xr'_{\nu-1}(x) \right] = \overline{R} \cdot \frac{x \left[ I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) \right]}{I^2_\nu(x)}.$$ 

Since $\overline{R} \in (0, 1)$ to prove $\Phi'_{2\nu}(x) < 1$ it would enough to show the Turán type inequality

$$I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{x} \cdot I^2_\nu(x). \quad (15)$$
For this observe that

$$
\Phi'_{2\nu}(x) = R \left[ r_{\nu-1}(x) + xr'_{\nu-1}(x) \right] = R \cdot \frac{x \left[ I^2_{\nu}(x) - I_{\nu-1}(x)I_{\nu+1}(x) \right]}{I^2_{\nu}(x)}.
$$

Since $R \in (0, 1)$ to prove $\Phi'_{2\nu}(x) < 1$ it would enough to show the Turán type inequality

$$
I^2_{\nu}(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{x} \cdot I^2_{\nu}(x).
$$

But, this inequality is valid for $\nu \geq \frac{1}{2}$ and $x > 0$, and it is equivalent to the fact that the function $x \mapsto x l'_{\nu}(x)/l_{\nu}(x) - x$ is strictly decreasing on $(0, \infty)$ for $\nu \geq \frac{1}{2}$, which was proved by Gronwall [Gron]. It is important to note here that Hamsici and Martinez [HM] used also the right-hand side of (15) for $1/(x + \nu)$ instead of $1/x$ in modeling the data of two spherical-homoscedastic von Mises-Fisher distribution. The correction of their result based on the right-hand side of (15) was made also by the author [Ba6].
Theorem

Let $\mu = \nu^2 - 1/4$. If $|\nu| \geq 1/2$ and $x > 0$, then the next Turán type inequalities are valid

$$- \frac{1}{x} \cdot K_\nu^2(x) \leq K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) \leq - \left(1 - \frac{\mu}{x^2}\right) \frac{1}{x} \cdot K_\nu^2(x). \quad (16)$$

Moreover, if $|\nu| < 1/2$ and $x > 0$, then the above inequalities are reversed, that is,

$$- \left(1 - \frac{\mu}{x^2}\right) \frac{1}{x} \cdot K_\nu^2(x) < K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) < - \frac{1}{x} \cdot K_\nu^2(x). \quad (17)$$

In (16) we have equality for $\nu = 1/2$. The left-hand side of (16) is sharp as $x \to 0$ when $1/2 \leq |\nu| \leq 1$, while (17) is sharp as $x \to 0$ for all $|\nu| < 1/2$. Each of the above inequalities are sharp as $x \to \infty$. 
Proof of inequalities (16) and (17)

Consider the corresponding Turánian \( \Delta_\nu(x) = K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) \).

Observe that, since \( K_\nu(x) = K_{-\nu}(x) \), the Turánian \( \Delta_\nu(x) \) is even in \( \nu \). Thus, it is enough to show the inequality (16) for \( \nu \geq 1/2 \) and the inequality (17) for \( \nu \in [0, 1/2) \).

Now, by using the recurrence relations

\[
K_{\nu-1}(x) = -\left(\frac{\nu}{x}\right)K_\nu(x) - K'_\nu(x)
\]

and

\[
K_{\nu+1}(x) = \left(\frac{\nu}{x}\right)K_\nu(x) - K'_\nu(x)
\]

clearly we have \( \Delta_\nu(x) = \left(1 + \frac{\nu^2}{x^2}\right)K_\nu^2(x) - [K'_\nu(x)]^2 \). On the other hand, recall that \( K_\nu \) is a particular solution of the second-order differential equation

\[
x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0,
\]

and this in turn implies that

\[
K''_\nu(x) = \left(1 + \frac{\nu^2}{x^2}\right)K_\nu(x) - \left(\frac{1}{x}\right)K'_\nu(x).
\]
Proof of inequalities (16) and (17)

Consequently, one has

\[ \Delta_\nu(x) = \frac{1}{x} K_\nu^2(x) \left[ \frac{xK'_\nu(x)}{K_\nu(x)} \right]' , \]

which in view of the recurrence relation

\[ xK'_\nu(x)/K_\nu(x) = -\nu - xK_{\nu-1}(x)/K_\nu(x) , \]

can be rewritten as

\[ \Delta_\nu(x) = -\frac{1}{x} K_\nu^2(x) \left\{ \frac{K_{\nu-1}(x)}{K_\nu(x)} + x \left[ \frac{K_{\nu-1}(x)}{K_\nu(x)} \right]' \right\} . \] (18)
Proof of inequalities (16) and (17)

Now, consider the integral representation

\[
\frac{K_{\nu-1}(\sqrt{x})}{\sqrt{x}K_{\nu}(\sqrt{x})} = \frac{4}{\pi^2} \int_0^\infty \frac{\gamma_{\nu}(t)dt}{x + t^2}, \quad \text{where} \quad \gamma_{\nu}(t) = \frac{t^{-1}}{J_{\nu}^2(t) + Y_{\nu}^2(t)}
\]

(19)

and \(x > 0, \nu \geq 0\). Here \(J_{\nu}\) and \(Y_{\nu}\) stand for the Bessel function of the first and second kinds. The above integral representation was given implicitly by Grosswald [Gros] and has been written in the form (19) by Ismail [Is]. Let \(\phi_{\nu} : (0, \infty) \rightarrow \mathbb{R}\) be defined by

\[
\phi_{\nu}(x) = \frac{\Delta_{\nu}(x)}{K_{\nu}^2(x)} = 1 - \frac{K_{\nu-1}(x)K_{\nu+1}(x)}{K_{\nu}^2(x)}.
\]

Then by using the relations (18) and (19) we have

\[
\phi_{\nu}(x) = -\frac{1}{x} \left[ \frac{4}{\pi^2} \int_0^\infty \frac{x\gamma_{\nu}(t)dt}{x^2 + t^2} + \frac{4}{\pi^2} \int_0^\infty \frac{x(t^2 - x^2)\gamma_{\nu}(t)dt}{(x^2 + t^2)^2} \right] = -\frac{4}{\pi^2} \int_0^\infty \frac{2t^2\gamma_{\nu}(t)dt}{(x^2 + t^2)^2}.
\]
Proof of inequalities (16) and (17)

On the other hand, it is known that [Wa, p. 446] the function \( t \mapsto 1/\gamma_\nu(t) \) is decreasing on \((0, \infty)\) for all \( \nu > 1/2 \) and is increasing on \((0, \infty)\) for all \( 0 \leq \nu < 1/2 \). Consequently, we obtain that \( \gamma_\nu(t) < \pi/2 \) for all \( t > 0 \) and \( \nu > 1/2 \). Moreover, \( \gamma_\nu(t) > \pi/2 \) for all \( t > 0 \) and \( 0 \leq \nu < 1/2 \). Thus, we have

\[
\phi_\nu(x) > -\frac{4}{\pi} \int_0^\infty \frac{t^2 dt}{(x^2 + t^2)^2} = -\frac{1}{x},
\]

where \( \nu > 1/2 \) and \( x > 0 \). The same proof works in the case \( 0 \leq \nu < 1/2 \). The only difference is that the above inequality is reversed. Now, by using for \( \nu = 1/2 \) the relations [Wa, p. 79]

\[
K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x), \quad K_\nu(x) = K_{-\nu}(x),
\]

we obtain \( \phi_{1/2}(x) = -1/x \). This completes the proof of the left-hand side of (16) and of the right-hand side of (17).
Proof of inequalities (16) and (17)

Observe that the inequality \([\text{Har1}, \text{eq. (4.6)}]\)

\[
t \left( 1 - \frac{\mu}{t^2} \right) \left[ J^2_\nu(t) + Y^2_\nu(t) \right] < \frac{2}{\pi},
\]

(20)

where \(t > 0\) and \(\nu > 1/2\), is equivalent to

\[
\gamma_\nu(t) > \left( 1 - \frac{\mu}{t^2} \right) \frac{\pi}{2}.
\]

(21)

Since

\[
t \left[ J^2_{1/2}(t) + Y^2_{1/2}(t) \right] = t \left[ \frac{2}{\pi t} \sin^2 t + \frac{2}{\pi t} \cos^2 t \right] = \frac{2}{\pi},
\]

for \(\nu = 1/2\) in inequalities (20) and (21) we have equality. These in turn imply that for all \(x > 0\) and \(\nu \geq 1/2\) we have

\[
\phi_\nu(x) \leq -\frac{4}{\pi} \int_0^\infty \frac{t^2 dt}{(x^2 + t^2)^2} + \frac{4\mu}{\pi} \int_0^\infty \frac{dt}{(x^2 + t^2)^2} = -\frac{1}{x} + \frac{\mu}{x^3},
\]

with equality when \(\nu = 1/2\), that is, \(\mu = 0\). The same proof works in the case \(0 \leq \nu < 1/2\). The only difference is that the inequality (20) is reversed, according to \([\text{Har1}, \text{eq. (4.7)}]\), and then (21) is reversed too. QED.
Theorem

If $\mu = \nu^2 - 1/4 \leq 0$ and $x > \sqrt{-\mu}$, then the next Turán type inequality is valid

\[
- \frac{4}{\pi} \left[ \arccos \left( \frac{\sqrt{-\mu}}{x} \right) \frac{\sqrt{-\mu}}{2x^2} + \frac{\sqrt{-\mu}}{x^2} \right] \cdot K_{2\nu}^2(x) \leq K_{\nu}^2(x) - K_{\nu-1}(x)K_{\nu+1}(x). \tag{22}
\]

In (22) we have equality for $\nu = 1/2$. The above inequality is sharp as $x \to \infty$.

Theorem

If $|\nu| \geq 1/2$ and $x > 0$, then the following Turán type inequality holds

\[
K_{2\nu}^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) \leq - \frac{1}{\sqrt{x^2 + \nu^2 - 1/4}} \cdot K_{\nu}^2(x). \tag{23}
\]

In (23) we have equality for $\nu = 1/2$. This inequality is sharp as $x \to \infty$. 
The graph of the functions $x \mapsto L(x) = -\frac{1}{x}$, $x \mapsto 1 - K_0(x)K_2(x)/K_1^2(x)$, $x \mapsto U_1(x) = -\frac{1}{x} \left(1 - \frac{3}{4x^2}\right)$ and $x \mapsto U_2(x) = -\frac{1}{\sqrt{x^2 + \frac{3}{4}}}$. 
By definition a function $f : [a, b] \subseteq \mathbb{R} \rightarrow (0, \infty)$ is log-convex if $\ln f$ is convex, i.e. if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}.$$ 

Similarly, a function $g : [a, b] \subseteq (0, \infty) \rightarrow (0, \infty)$ is said to be geometrically (or multiplicatively) convex if $g$ is convex with respect to the geometric mean, i.e. if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$g \left( x^{\lambda} y^{1-\lambda} \right) \leq [g(x)]^\lambda [g(y)]^{1-\lambda}.$$
We note that if the functions $f$ and $g$ are differentiable then $f$ is (strictly) log-convex if and only if the function $x \mapsto f'(x)/f(x)$ is (strictly) increasing on $[a, b]$, while $g$ is (strictly) geometrically convex if and only if the function $x \mapsto xg'(x)/g(x)$ is (strictly) increasing on $[a, b]$. A similar definition and characterization of differentiable (strictly) log-concave and (strictly) geometrically concave functions also holds. Observe that the left-hand side of (1) together with (5), and the right-hand side of (2) together with (8) imply that $I_\nu$ is strictly geometrically convex on $(0, \infty)$ for all $\nu > -1$, while $K_\nu$ is strictly geometrically concave on $(0, \infty)$ for all $\nu \in \mathbb{R}$, respectively.
Moreover, summing up the corresponding parts of the right-hand sides of Turán type inequalities (9) and (23) and taking into account the relations (5) and (8) we obtain

\[
\left[ \frac{xP'_\nu(x)}{P_\nu(x)} \right]' = \left[ \frac{xI'_\nu(x)}{I_\nu(x)} \right]' + \left[ \frac{xK'_\nu(x)}{K_\nu(x)} \right]' < 0
\]  

(24)

for all \( \nu \geq 1/2 \) and \( x > 0 \).

Consequently, the following result is valid.

**Corollary**

If \( \nu \geq 1/2 \), then the function \( P_\nu \) is strictly geometrically concave on \((0, \infty)\). In particular, for all \( x, y > 0 \) and \( \nu \geq 1/2 \) we have

\[
P_\nu(\sqrt{xy}) > \sqrt{P_\nu(x)P_\nu(y)}.
\]
It is also important to note here that since for
\[ \omega_\nu(x) = xP_\nu(x) = xl_\nu(x)K_\nu(x) \] we have
\[ \frac{x\omega_\nu'(x)}{\omega_\nu(x)} = 1 + \frac{xP_\nu'(x)}{P_\nu(x)}, \]
the above result implies that the function \( \omega_\nu \) is also strictly geometrically concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). On the other hand, since the function \( 2\omega_\nu \) is a continuous cumulative distribution function, according to Hartman and Watson [HW], it follows that the \( \omega_\nu \) is strictly log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). This results is similar to the result of Hartman [Har2], who proved that \( \omega_\nu \) is strictly concave on \((0, \infty)\) for all \( \nu > 1/2 \). Since \( x \mapsto 2\omega_{1/2}(x) = 1 - e^{-2x} \) is strictly concave on \((0, \infty)\), we conclude that in fact the function \( \omega_\nu \) is strictly concave, and hence strictly log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \).
We also mention here that recently it was shown (see [Ba4]) that surprisingly the most common continuous univariate distributions, like the standard normal, standard log-normal (or Gibrat), Student’s $t$, Weibull (or Rosin-Rammler), Kumaraswamy, Fisher-Snedecor’s $F$, gamma and Sichel (or generalized inverse Gaussian) distributions, have the property that their probability density functions are geometrically concave and consequently their cumulative distribution functions and survival functions are also geometrically concave. Taking into account the above discussion, the distribution of which cumulative distribution function $2\omega_\nu$ was considered by Hartman and Watson [HW] belongs also to the class of geometrically concave univariate distributions.
Turán’s inequality for Legendre polynomials
Turán type inequalities for modified Bessel functions

References


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